Figures of lecture 3

Global quantities and the first law

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https://relativite.obspm.fr/blackholes/aei23/

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These lectures

provide an introduction to BH thermodynamics

- focussing on classical (non-quantum) aspects
- keeping the spacetime dimension n general
- not restricting the theory of gravity to general relativity

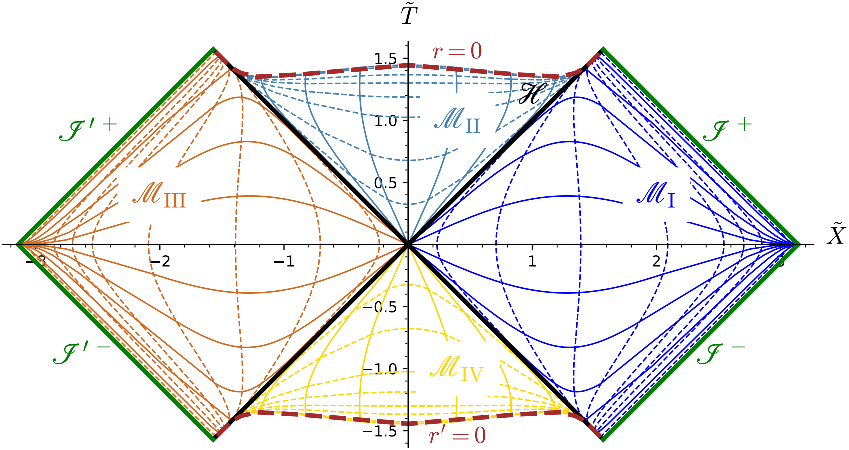
Home page

https://relativite.obspm.fr/blackholes/aei23

includes

- the lecture notes (draft)
- some SageMath notebooks
- these slides

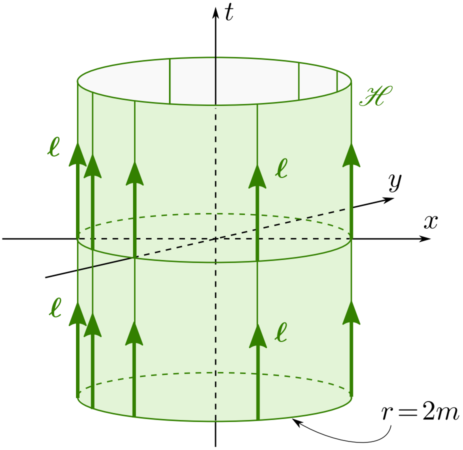
Orbits of stationary action in Schwarzschild spacetime



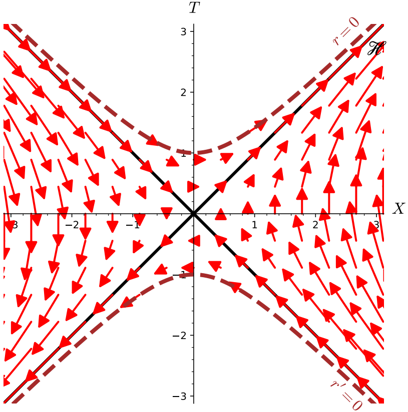
Maximal extension of Schwarzschild spacetime represented in Penrose-Frolov-Novikov coordinates

Dashed lines = orbits of stationary action (field lines of Killing vector ξ)

Example of ξ null on $\mathscr{H}$: the Schwarzschild horizon

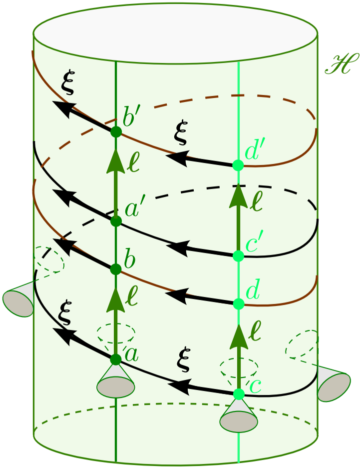


$$\xi \overset{\mathscr{H}}{=} \ell$$

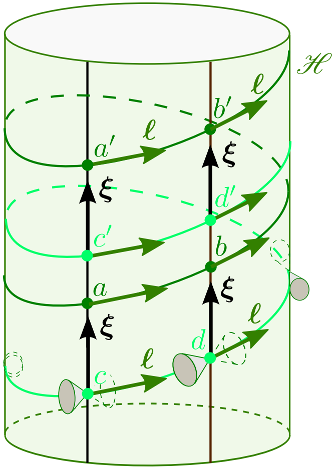


ξ in all spacetime
(Kruskal-Szekeres coordinates)

Two equivalent views of $\mathscr{H}$ when ξ is spacelike on it



null geodesic generators
drawn vertically



field lines of Killing vector ξ
drawn vertically

Example: area of the Kerr black hole

Use Kerr coordinates $(\tilde{t}, r, \theta, \tilde{\varphi})$, which are regular on $\mathscr{H}$, instead of the Boyer-Lindquist coordinates.
As a non-expanding horizon, $\mathscr{H}$ has a well-defined (cross-section independent) area A:

$$A = \int_{\mathscr{S}} \sqrt{q}\, \mathrm{d}\theta\, \mathrm{d}\tilde{\varphi}$$

- $\mathscr{S}$: cross-section of $\mathscr{H}$ defined in terms of Kerr coordinates by
$$\begin{cases} \tilde{t} = \tilde{t}_0 \\ r = r_+ \end{cases}$$
$\Longrightarrow$ coordinates spanning $\mathscr{S}$: $y^a = (\theta, \tilde{\varphi})$
- $q := \det(q_{ab})$, with q_{ab} components of the Riemannian metric $\boldsymbol{q}$ induced on $\mathscr{S}$ by the spacetime metric $\boldsymbol{g}$

Example: area of the Kerr black hole

Kerr metric $\boldsymbol{g}$ in terms of the Kerr coordinates $(\tilde{t}, r, \theta, \tilde{\varphi})$:

$$
\begin{aligned}
g_{\mu\nu}\, dx^\mu dx^\nu = \; & -\left(1 - \frac{2mr}{\rho^2}\right) d\tilde{t}^2 + \frac{4mr}{\rho^2} d\tilde{t}\, dr - \frac{4amr\sin^2\theta}{\rho^2}\, d\tilde{t}\, d\tilde{\varphi} \\
& + \left(1 + \frac{2mr}{\rho^2}\right) dr^2 - 2a\left(1 + \frac{2mr}{\rho^2}\right)\sin^2\theta\, dr\, d\tilde{\varphi} \\
& + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2a^2 mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\tilde{\varphi}^2 .
\end{aligned}
$$

The metric $\boldsymbol{q}$ induced in $\mathscr{S}$ is obtained by setting $d\tilde{t} = 0$, $dr = 0$, and $r = r_+$:

$$
q_{ab}\, dy^a dy^b = (r_+^2 + a^2\cos^2\theta)\, d\theta^2 + \left(r_+^2 + a^2 + \frac{2a^2 m r_+ \sin^2\theta}{r_+^2 + a^2\cos^2\theta}\right)\sin^2\theta\, d\tilde{\varphi}^2
$$

Example: area of the Kerr black hole

r_+ is a zero of $\Delta := r^2 - 2mr + a^2 \Longrightarrow 2mr_+ = r_+^2 + a^2$

$\Longrightarrow q_{ab}$ can be rewritten as

$$q_{ab}\, \mathrm{d}y^a \mathrm{d}y^b = (r_+^2 + a^2\cos^2\theta)\, \mathrm{d}\theta^2 + \frac{(r_+^2 + a^2)^2}{r_+^2 + a^2\cos^2\theta}\ \sin^2\theta\, \mathrm{d}\tilde{\varphi}^2$$

$$\Longrightarrow q := \det(q_{ab}) = (r_+^2 + a^2)^2 \sin^2\theta$$

$$\Longrightarrow A = (r_+^2 + a^2) \underbrace{\int_{\mathscr{S}} \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\tilde{\varphi}}_{4\pi}$$

$$\Longrightarrow \boxed{A = 4\pi(r_+^2 + a^2) = 8\pi m r_+}$$

Since $r_+ := m + \sqrt{m^2 - a^2}$, we get

$$A = 8\pi m(m + \sqrt{m^2 - a^2})$$