Geometry of Killing horizons and applications to black hole physics

1. Null hypersurfaces and non-expanding horizons

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
CNRS, Observatoire de Paris-PSL, Université Paris Cité, Meudon, France

https://relativite.obspm.fr/blackholes/ihp24/

Quantum and classical fields interacting with geometry
Institut Henri Poincaré, Paris, France
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1. Null hypersurfaces and non-expanding horizons *(today)*
2. Killing horizons *(today)*
3. Stationary black holes *(tomorrow)*
4. Degenerate Killing horizons and their near-horizon geometry *(tomorrow)*
5. Exploring the extremal Kerr near-horizon geometry with SageMath *(on Thursday)*

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Prerequisite

An introductory course on general relativity

Home page for the lectures

https://relativite.obspm.fr/blackholes/ihp24/

includes

- these slides
- the lecture notes (draft)
- some SageMath notebooks

Lecture 1: Null hypersurfaces and non-expanding horizons

Outline

Framework of the lectures

spacetime = $(\mathscr{M}, \boldsymbol{g})$

- $\mathscr{M}$: n-dimensional smooth manifold ($n \geq 3$)
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$

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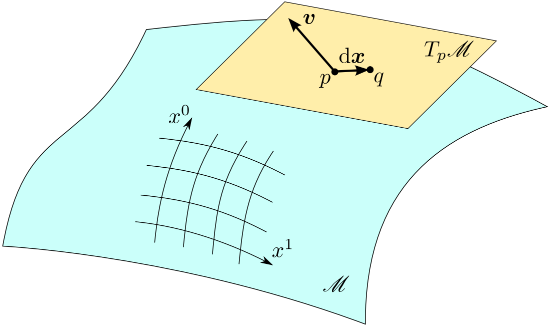

Smooth manifold: topological space $\mathscr{M}$ that *locally* resembles $\mathbb{R}^n$ (but maybe not globally)

$\Longrightarrow$ **coordinate charts**

$\Longrightarrow$ **tangent vectors**

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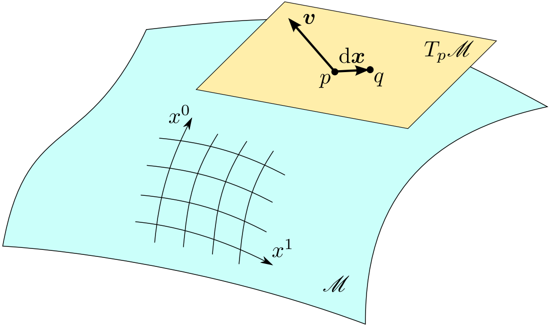

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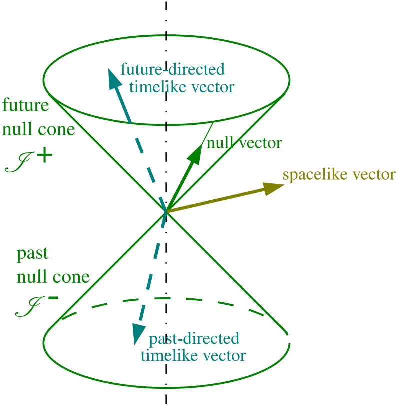
$\Longrightarrow$ **coordinate charts**
$\Longrightarrow$ **tangent vectors**

Remark: vector connecting two points p and q defined only for p and q infinitely close

Metric's null cone



Vector $\boldsymbol{v} \in T_p\mathscr{M}$ is

- **spacelike** $\iff \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$
- **null** $\iff \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$
- **timelike** $\iff \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$

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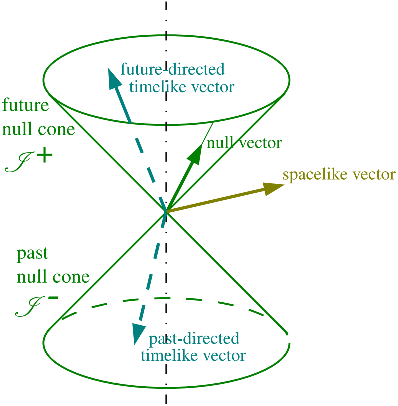

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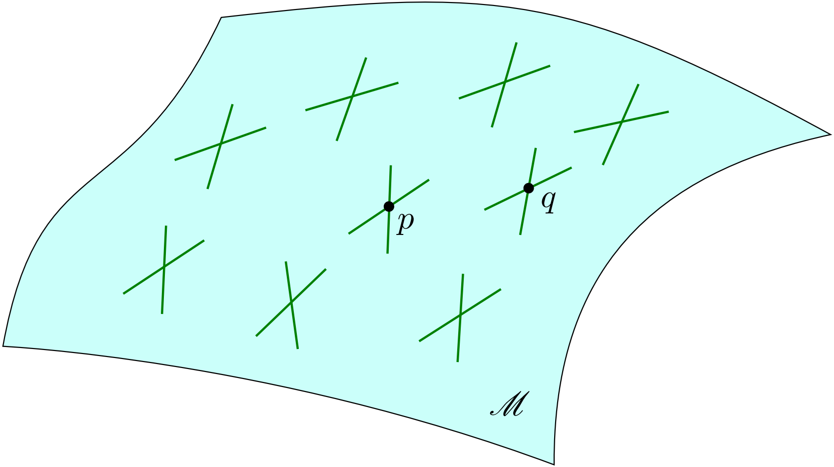
Additional assumption:
the spacetime $(\mathscr{M}, \boldsymbol{g})$ is **time-oriented**
$\Longrightarrow$ future and past directions

Lorentzian manifold $(\mathscr{M}, \boldsymbol{g})$



Einstein's equation

$(\mathscr{M}, \boldsymbol{g})$ is ruled by general relativity $\iff$ $\boldsymbol{g}$ obeys **Einstein's equation:**

$$\boxed{\boldsymbol{R} - \frac{1}{2} R\, \boldsymbol{g} + \Lambda\, \boldsymbol{g} = 8\pi \boldsymbol{T}}$$

where

- $\boldsymbol{R} := \mathrm{Ric}(\boldsymbol{g})$, Ricci tensor: $R_{\alpha\beta} = \mathrm{Riem}(\boldsymbol{g})^{\mu}{}_{\alpha\mu\beta}$
- $R := g^{\mu\nu} R_{\mu\nu}$, Ricci scalar
- Λ cosmological constant
- $\boldsymbol{T}$ energy-momentum tensor of matter/fields

In these lectures: $\Lambda = 0$.

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We shall make clear whether a black hole property relies on Einstein's equation or not.

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Hypersurfaces in spacetime

A **hypersurface** of the n-dimensional spacetime $(\mathscr{M}, \boldsymbol{g})$ is an embedded submanifold of $\mathscr{M}$ of dimension $n-1$ (codimension 1).

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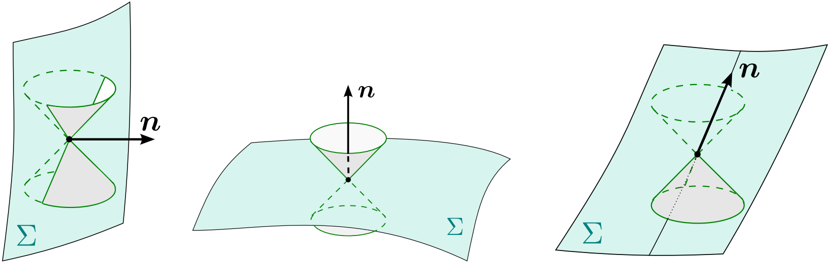
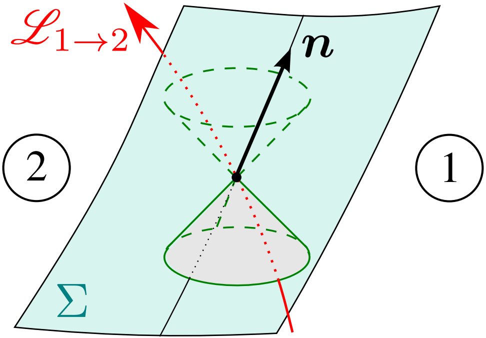
Locally, a hypersurface Σ can be of one of 3 types ($\boldsymbol{n}$ = normal to Σ):



Σ timelike	Σ spacelike	Σ null
$\boldsymbol{g}\|_{\Sigma}$ Lorentzian	$\boldsymbol{g}\|_{\Sigma}$ Riemannian	$\boldsymbol{g}\|_{\Sigma}$ degenerate
$\boldsymbol{n}$ spacelike	$\boldsymbol{n}$ timelike	$\boldsymbol{n}$ null (and tangent to Σ)

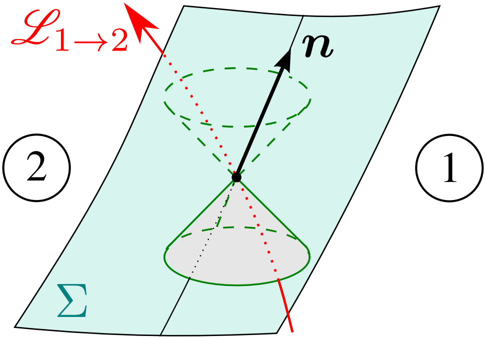
Null hypersurface as a causal boundary



For timelike worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...

Null hypersurface as a causal boundary



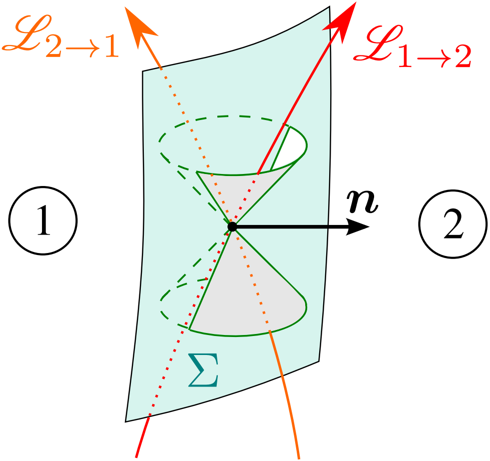
For timelike worldlines $\mathscr{L}$ directed towards the future:

null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...
...and elected! (as a consequence of the formal definition of a black hole)

Theorem (Penrose 1968)

Wherever it is smooth, the event horizon of a black hole is a null hypersurface.

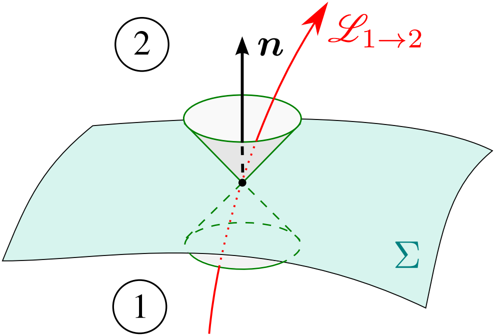
Timelike hypersurfaces are not causal boundaries



For timelike worldlines $\mathscr{L}$ directed towards the future:

timelike hypersurface = 2-way membrane
$\Longrightarrow$ not eligible for a black hole boundary

Spacelike hypersurfaces



For timelike worldlines $\mathscr{L}$ directed towards the future:

spacelike hypersurface = 1-way membrane
$\Longrightarrow$ in the dynamical black hole context: trapping horizons = spacelike hypersurfaces

Normal to a null hypersurface

A *generic* hypersurface $\mathscr{H}$ of $\mathscr{M}$ can be (locally) defined as a level set (or isosurface) of some scalar field $u : \mathscr{M} \to \mathbb{R}$ such that $\mathbf{d}u \neq 0$:

$$\mathscr{H} = \{p \in \mathscr{M}, u(p) = 0\}$$

[1] If necessary, consider $u' := -u$ instead of u

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Any vector field $\boldsymbol{\ell}$ normal to $\mathscr{H}$ must be collinear to the gradient of u:

$$\boxed{\boldsymbol{\ell} = -\mathrm{e}^{\rho}\, \vec{\boldsymbol{\nabla}} u}$$

where ρ is some scalar field and the minus sign is chosen for convenience. In term of components with respect to a coordinate system (x^α):

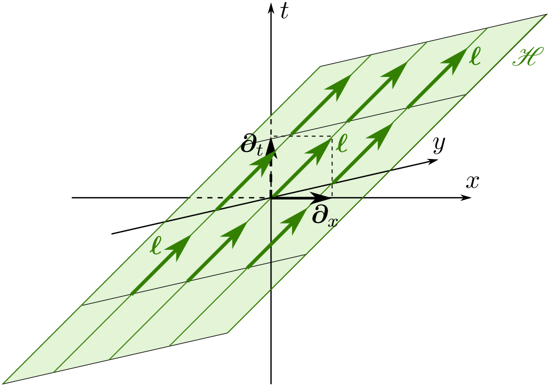
$$\ell^\alpha = -\mathrm{e}^\rho \nabla^\alpha u = -\mathrm{e}^\rho g^{\alpha\mu} \nabla_\mu u = -\mathrm{e}^\rho g^{\alpha\mu} \partial_\mu u$$

$\mathscr{H}$ **null** hypersurface $\iff$ $\boldsymbol{g}(\boldsymbol{\ell}, \boldsymbol{\ell}) = 0$ $\iff$ $g^{\mu\nu} \partial_\mu u\, \partial_\nu u = 0$

Assumption: $\boldsymbol{\ell}$ is future-directed[1]

[1] If necessary, consider $u' := -u$ instead of u

Example 1: null hyperplane in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$$\boxed{u := t - x = 0}$$

$$\boldsymbol{\nabla} u = \mathbf{d}t - \mathbf{d}x$$

$$\nabla_\alpha u = (1, -1, 0, 0)$$

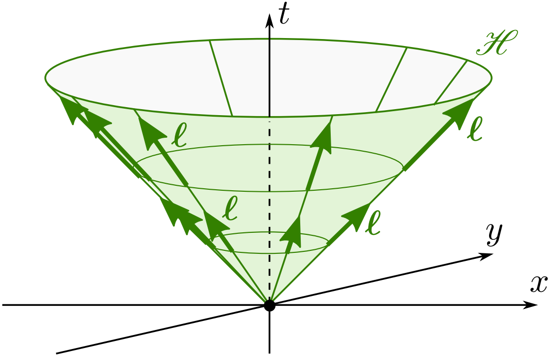
$$\nabla^\alpha u = (-1, -1, 0, 0)$$

Choose $\rho = 0$

$$\Longrightarrow \ell^\alpha = (1, 1, 0, 0)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x}$$

Example 2: future null cone in Minkowski spacetime



$$\boldsymbol{g} = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2$$

$$\boxed{u := t - \sqrt{x^2 + y^2 + z^2} = 0}$$

$$\boldsymbol{\nabla} u = \mathbf{d}t - \frac{x}{r}\mathbf{d}x - \frac{y}{r}\mathbf{d}y - \frac{z}{r}\mathbf{d}z$$

$$r := \sqrt{x^2 + y^2 + z^2}$$

$$\nabla_\alpha u = \left(1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

$$\nabla^\alpha u = \left(-1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

Choose $\rho = 0$

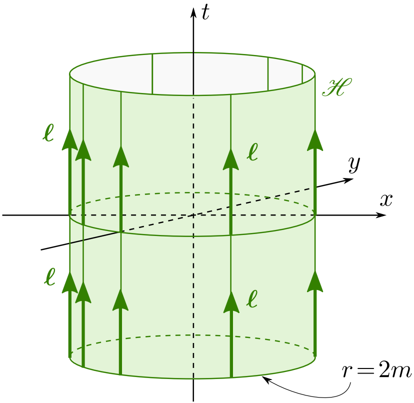
$$\Longrightarrow \ell^\alpha = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{x}{r}\boldsymbol{\partial}_x + \frac{y}{r}\boldsymbol{\partial}_y + \frac{z}{r}\boldsymbol{\partial}_z}$$

Example 3: Schwarzschild horizon

in Eddington-Finkelstein coordinates

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right)\mathbf{d}t^2 + \frac{4m}{r}\,\mathbf{d}t\,\mathbf{d}r + \left(1 + \frac{2m}{r}\right)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2\theta\,\mathbf{d}\varphi^2$$



$$u := \left(1 - \frac{r}{2m}\right)\exp\left(\frac{r-t}{4m}\right) = 0$$

$$\mathscr{H}: \quad u = 0 \iff r = 2m$$

$$\boldsymbol{\nabla} u = \frac{1}{4m}\mathrm{e}^{(r-t)/(4m)}\left[-\left(1 - \frac{r}{2m}\right)\mathbf{d}t - \left(1 + \frac{r}{2m}\right)\mathbf{d}r\right]$$

Exercise: compute $\boldsymbol{\ell}$ with ρ chosen so that $\ell^t = 1$ and get

$$\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{r-2m}{r+2m}\,\boldsymbol{\partial}_r \implies \boldsymbol{\ell} \stackrel{\mathscr{H}}{=} \boldsymbol{\partial}_t$$

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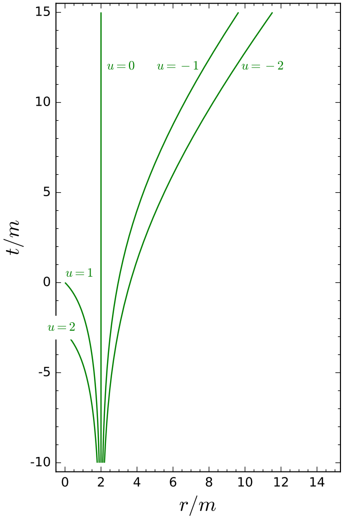

Hypersurfaces of constant value of u around the Schwarzschild horizon $u=0$

Frobenius identity

A fundamental identity obeyed by any normal $\boldsymbol{\ell}$ to a hypersurface

Starting point: $\boldsymbol{\ell} = -\mathrm{e}^{\rho}\, \vec{\boldsymbol{\nabla}} u$

$$\Longrightarrow \quad \ell_\alpha = -\mathrm{e}^{\rho}\, \nabla_\alpha u$$

$$\Longrightarrow \quad \nabla_\alpha \ell_\beta = -\mathrm{e}^{\rho} \nabla_\alpha \rho \nabla_\beta u - \mathrm{e}^{\rho} \nabla_\alpha \nabla_\beta u$$

$$\Longrightarrow \quad \nabla_\alpha \ell_\beta - \nabla_\beta \ell_\alpha = -\mathrm{e}^{\rho} \nabla_\alpha \rho \nabla_\beta u + \mathrm{e}^{\rho} \nabla_\beta \rho \nabla_\alpha u$$

$$\Longrightarrow \quad \boxed{\nabla_\alpha \ell_\beta - \nabla_\beta \ell_\alpha = \nabla_\alpha \rho\, \ell_\beta - \nabla_\beta \rho\, \ell_\alpha}$$

In terms of exterior (Cartan) calculus:

$$\boxed{\mathbf{d}\underline{\boldsymbol{\ell}} = \mathbf{d}\rho \wedge \underline{\boldsymbol{\ell}}}$$

where

- $\underline{\boldsymbol{\ell}}$ is the 1-form metric-dual to vector $\boldsymbol{\ell}$: $\underline{\boldsymbol{\ell}} = \ell_\alpha \mathbf{d}x^\alpha$, $\ell_\alpha = g_{\alpha\mu}\ell^\mu$
- $\mathbf{d}\underline{\boldsymbol{\ell}}$ is the exterior derivative of $\underline{\boldsymbol{\ell}}$ (2-form)
- $\wedge$ is the exterior product of p-forms

Null geodesic generators

Contract Frobenius identity with $\boldsymbol{\ell}$:

$$\ell^\mu \nabla_\mu \ell_\alpha - \ell^\mu \nabla_\alpha \ell_\mu = \ell^\mu \nabla_\mu \rho \, \ell_\alpha - \underbrace{\ell^\mu \ell_\mu}_{0} \nabla_\alpha \rho$$

Now $\ell^\mu \nabla_\alpha \ell_\mu = \nabla_\alpha (\underbrace{\ell^\mu \ell_\mu}_{0}) - \ell_\mu \nabla_\alpha \ell^\mu \Longrightarrow \ell^\mu \nabla_\alpha \ell_\mu = 0$

Hence

$$\ell^\mu \nabla_\mu \ell_\alpha = \kappa \, \ell_\alpha \quad \text{with } \kappa := \ell^\mu \nabla_\mu \rho = \boldsymbol{\nabla}_{\boldsymbol{\ell}} \, \rho$$

or, by metric duality (index raising):

$$\ell^\mu \nabla_\mu \ell^\alpha = \kappa \, \ell^\alpha$$

i.e.

$$\boxed{\boldsymbol{\nabla}_{\boldsymbol{\ell}} \, \boldsymbol{\ell} = \kappa \, \boldsymbol{\ell}}$$

Null geodesic generators

$\boxed{\nabla_{\boldsymbol{\ell}} \boldsymbol{\ell} = \kappa \boldsymbol{\ell}}$ $\Longrightarrow$ $\boldsymbol{\ell}$ is a **pregeodesic vector**, i.e. $\exists$ rescaling factor α such that $\boldsymbol{\ell}' = \alpha \boldsymbol{\ell}$ is a **geodesic vector**: $\nabla_{\boldsymbol{\ell}'} \boldsymbol{\ell}' = 0$

Exercise: prove it!

$\Longrightarrow$ the field lines of $\boldsymbol{\ell}$ are (null) geodesics.

κ is called the **non-affinity coefficient** of the null normal $\boldsymbol{\ell}$ because

$$\kappa = 0 \iff \lambda \text{ is an affine parameter}$$

where λ is the parameter along a geodesic field line of $\boldsymbol{\ell}$ whose derivative vector is $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda}$$

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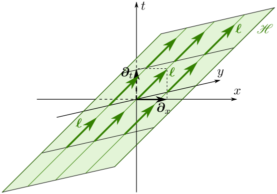
Any null hypersurface $\mathscr{H}$ is ruled by a family of null geodesics, called the **generators of** $\mathscr{H}$, and each vector field $\boldsymbol{\ell}$ normal to $\mathscr{H}$ is tangent to these null geodesics.

Examples of null geodesic generators

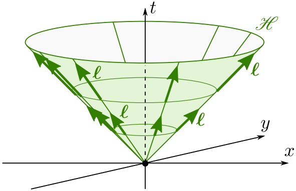
null hyperplane



$$\nabla_{\ell}\,\ell = 0$$

$$\kappa = 0$$

future null cone



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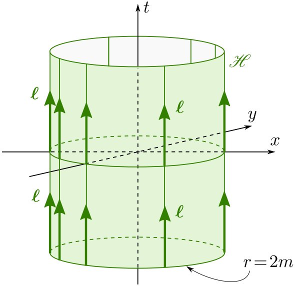
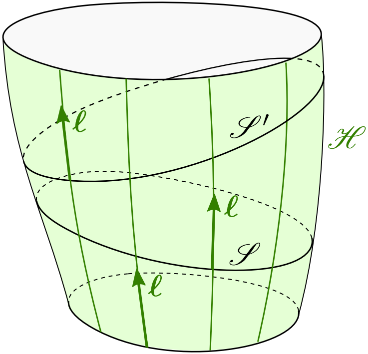
Schwarzschild horizon



$$\nabla_{\ell}\,\ell = \kappa\,\ell$$

$$\kappa = \frac{1}{4m}$$

Cross-sections of a null hypersurface



cross-section of the null hypersurface $\mathscr{H}$: $(n-2)$-dimensional submanifold $\mathscr{S} \subset \mathscr{H}$ such that

1. the null normal $\boldsymbol{\ell}$ is nowhere tangent to $\mathscr{S}$
2. each null geodesic generator of $\mathscr{H}$ intersects $\mathscr{S}$ at most once

complete cross-section: each null geodesic generator of $\mathscr{H}$ intersects $\mathscr{S}$ exactly once

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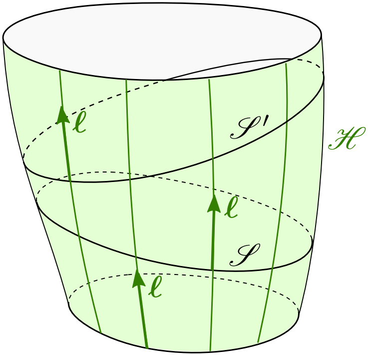

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Any cross-section $\mathscr{S}$ is spacelike, i.e. all vectors tangent to $\mathscr{S}$ are spacelike.

Proof: a vector tangent to $\mathscr{H}$ cannot be timelike, nor null and not normal.

Induced metric on a cross-section

Induced metric $\boldsymbol{q}$ on a cross-section $\mathscr{S}$:

$$\forall (\boldsymbol{u}, \boldsymbol{v}) \in T_p\mathscr{S} \times T_p\mathscr{S}, \quad \boldsymbol{q}(\boldsymbol{u}, \boldsymbol{v}) := \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v})$$

$\mathscr{S}$ spacelike $\Longrightarrow$ $\boldsymbol{q}$ positive definite $\Longrightarrow$ $(\mathscr{S}, \boldsymbol{q})$ Riemannian manifold

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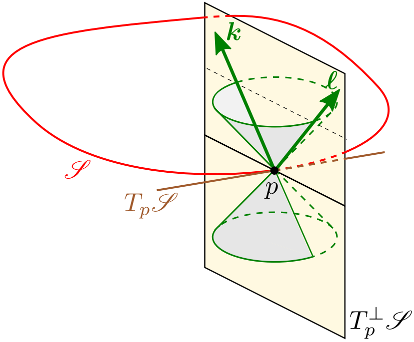

Orthogonal complement to $T_p\mathscr{S}$:
$\boldsymbol{q}$ Riemannian $\Longrightarrow$ $T_p^{\perp}\mathscr{S}$ timelike 2-plane s.t.

$$T_p\mathscr{M} = T_p\mathscr{S} \oplus T_p^{\perp}\mathscr{S}$$

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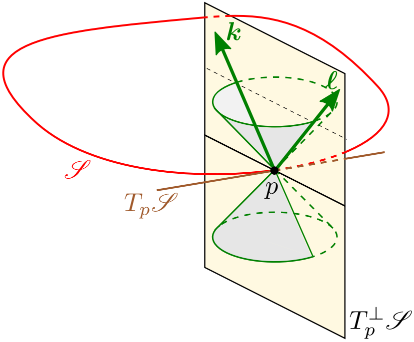

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Complement null normal to $\mathscr{S}$: null vector $\boldsymbol{k}$ s.t $\boldsymbol{k} \cdot \boldsymbol{\ell} = -1$ and

$$T_p^\perp\mathscr{S} = \mathrm{Span}\,(\boldsymbol{\ell}, \boldsymbol{k})$$

Extension of $\boldsymbol{q}$ to a spacetime tensor

- Extension to a bilinear form:

$$\boxed{\boldsymbol{q} := \boldsymbol{g} + \underline{\boldsymbol{\ell}} \otimes \underline{\boldsymbol{k}} + \underline{\boldsymbol{k}} \otimes \underline{\boldsymbol{\ell}}} \iff q_{\alpha\beta} := g_{\alpha\beta} + \ell_\alpha k_\beta + k_\alpha \ell_\beta$$

$$\forall (\boldsymbol{u}, \boldsymbol{v}) \in T_p\mathscr{M} \times T_p\mathscr{M}, \quad \boldsymbol{q}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{q}_{\mathscr{S}}(\boldsymbol{u}^{\parallel}, \boldsymbol{v}^{\parallel})$$

- **Orthogonal projector** onto $\mathscr{S}$:

$$\boxed{\vec{\boldsymbol{q}} := \mathrm{Id} + \boldsymbol{\ell} \otimes \underline{\boldsymbol{k}} + \boldsymbol{k} \otimes \underline{\boldsymbol{\ell}}} \iff q^\alpha{}_\beta := \delta^\alpha{}_\beta + \ell^\alpha\, k_\beta + k^\alpha\, \ell_\beta$$

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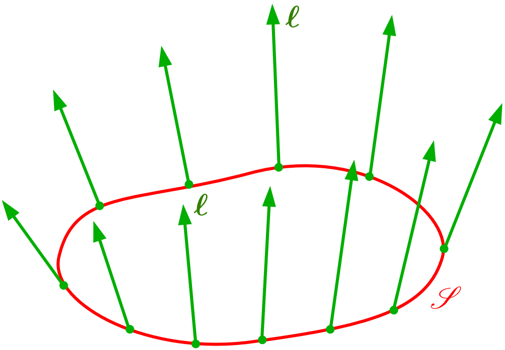
Remark: $\mathscr{H}$ being a null hypersurface, there is no orthogonal projector onto $\mathscr{H}$. Instead, on $\mathscr{S}$, one may introduce the **projector $\boldsymbol{\Pi}$ onto $\mathscr{H}$ along $\boldsymbol{k}$** by

$$\Pi^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + k^{\alpha}\ell_\beta$$

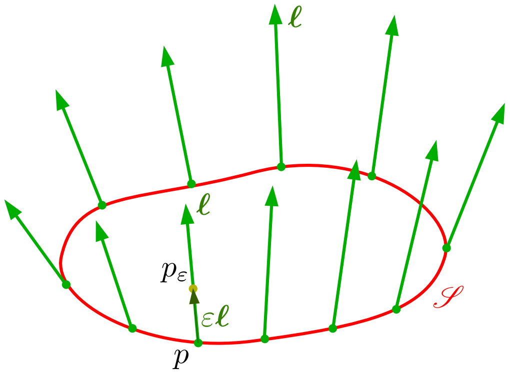
It fulfills $\boldsymbol{\Pi}(\boldsymbol{k}) = 0$ and $\forall \boldsymbol{v} \in T_p\mathscr{H}$, $\boldsymbol{\Pi}(\boldsymbol{v}) = \boldsymbol{v}$.

Expansion along a null normal



1. Consider a cross-section $\mathscr{S}$ and a null normal $\boldsymbol{\ell}$ to $\mathscr{H}$

Expansion along a null normal



1. Consider a cross-section $\mathscr{S}$ and a null normal $\boldsymbol{\ell}$ to $\mathscr{H}$
2. ε being a small parameter, displace the point p by the vector $\varepsilon\boldsymbol{\ell}$ to the point p_ε

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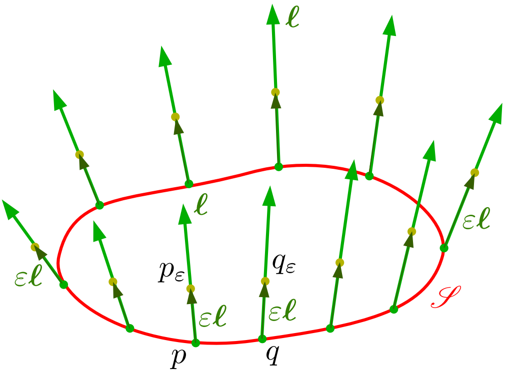

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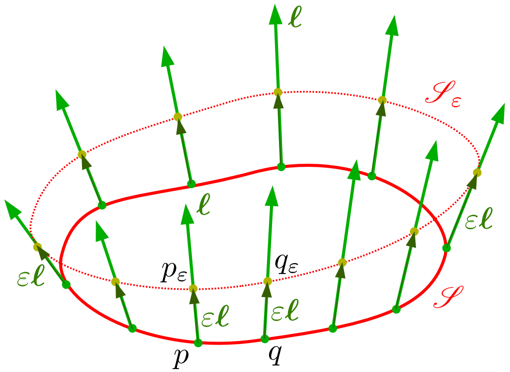

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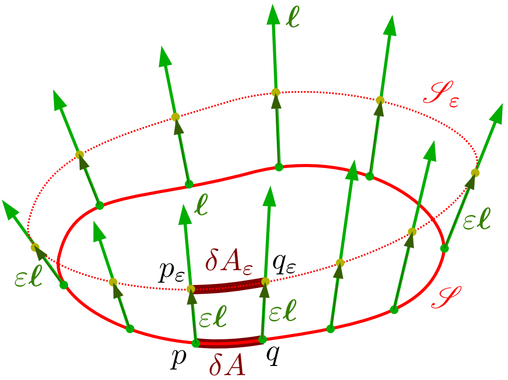

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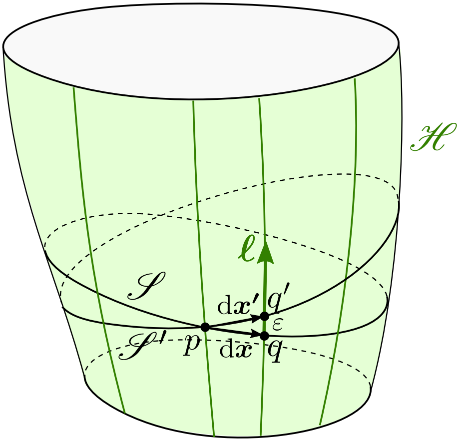
The **expansion along** $\boldsymbol{\ell}$ is defined from the relative change of the area δA (w.r.t. metric $\boldsymbol{q}$) of a surface element δS of $\mathscr{S}$ around p:

$$\theta_{(\boldsymbol{\ell})} := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A} = \mathcal{L}_{\boldsymbol{\ell}} \ln\sqrt{q} = q^{\mu\nu}\nabla_\mu \ell_\nu$$

Expansion along a null normal



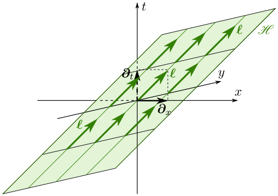
The expansion $\theta_{(\boldsymbol{\ell})}$ at a point $p \in \mathscr{H}$ depends solely on the null normal $\boldsymbol{\ell}$, not on the choice of the cross-section $\mathscr{S}$ through p.

$$\theta_{(\boldsymbol{\ell})} = q^{\mu\nu}\nabla_\mu \ell_\nu \Rightarrow \boxed{\theta_{(\boldsymbol{\ell})} = \nabla_\mu \ell^\mu - \kappa}$$

Remark: $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell} \Longrightarrow \theta_{(\boldsymbol{\ell}')} = \alpha\theta_{(\boldsymbol{\ell})}$

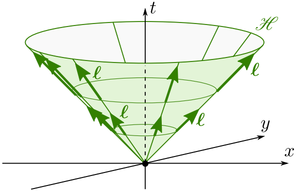
Examples of expansions

null hyperplane



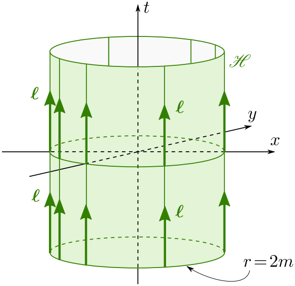
$$\theta_{(\boldsymbol{\ell})} = 0$$

future null cone



$$\theta_{(\boldsymbol{\ell})} = \frac{2}{r}$$

Schwarzschild horizon



$$\theta_{(\boldsymbol{\ell})} = 0$$

Deformation rate

Variation of $\mathscr{S}$'s metric $\boldsymbol{q}$ when evolved (Lie-dragged) along the null normal $\boldsymbol{\ell}$ $\Longrightarrow$ **deformation rate of $\mathscr{S}$ along $\boldsymbol{\ell}$**:

$$\boxed{\boldsymbol{\Theta} := \frac{1}{2}\vec{\boldsymbol{q}}^{*}\boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\,\boldsymbol{q}} \iff \Theta_{\alpha\beta} = \frac{1}{2}q^{\mu}{}_{\alpha}q^{\nu}{}_{\beta}\mathcal{L}_{\boldsymbol{\ell}}\,q_{\mu\nu}$$

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One has $\Theta_{\alpha\beta} = q^{\mu}{}_{\alpha} q^{\nu}{}_{\beta} \nabla_{\mu} \ell_{\nu}$, from which

$$\boxed{\nabla_{\alpha}\ell_{\beta} = \Theta_{\alpha\beta} + \omega_{\alpha}\ell_{\beta} - \ell_{\alpha} k^{\mu} \nabla_{\mu} \ell_{\beta},}$$

where $\boldsymbol{\omega}$ is the 1-form defined by

$$\omega_{\alpha} := -k^{\mu}\nabla_{\nu}\ell_{\mu}\Pi^{\nu}{}_{\alpha} = -k^{\mu}\nabla_{\alpha}\ell_{\mu} - k^{\mu}k^{\nu}\nabla_{\mu}\ell_{\nu}\,\ell_{\alpha}\ ,$$

$\Pi^{\nu}{}_{\alpha} := \delta^{\nu}{}_{\alpha} + k^{\nu}\ell_{\alpha}$ being the projector onto $\mathscr{H}$ along $\boldsymbol{k}$ introduced above.

Shear tensor

The trace-free part of $\boldsymbol{\Theta}$ is called the **shear tensor of $\mathscr{S}$ along $\boldsymbol{\ell}$**:

$$\boxed{\boldsymbol{\sigma} := \boldsymbol{\Theta} - \frac{1}{n-2}\theta_{(\boldsymbol{\ell})}\, \boldsymbol{q}} \iff \sigma_{\alpha\beta} = \Theta_{\alpha\beta} - \frac{1}{n-2}\theta_{(\boldsymbol{\ell})}\, q_{\alpha\beta}$$

By construction, ${\sigma^\mu}_\mu = 0$.

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By construction, $\sigma^{\mu}{}_{\mu} = 0$.

Remark 1: the tensor fields $\boldsymbol{q}$, $\boldsymbol{\Theta}$ and $\boldsymbol{\sigma}$ are tangent to $\mathscr{S}$ is the sense that

$$\forall \boldsymbol{v} \in T_p^{\perp}\mathscr{S}, \quad \boldsymbol{q}(\boldsymbol{v}, .) = \boldsymbol{\Theta}(\boldsymbol{v}, .) = \boldsymbol{\sigma}(\boldsymbol{v}, .) = 0$$

Remark 2: upon a change of null normal, the following rescaling holds:

$$\boldsymbol{\ell}' = \alpha\boldsymbol{\ell} \implies \theta_{(\boldsymbol{\ell}')} = \alpha\theta_{(\boldsymbol{\ell})}, \quad \boldsymbol{\Theta}' = \alpha\boldsymbol{\Theta}, \quad \boldsymbol{\sigma}' = \alpha\boldsymbol{\sigma}$$

Evolution of the expansion

Null Raychaudhuri equation

$$\nabla_{\boldsymbol{\ell}}\, \theta_{(\boldsymbol{\ell})} = \kappa \theta_{(\boldsymbol{\ell})} - \frac{1}{n-2}\, \theta^2_{(\boldsymbol{\ell})} - \sigma_{ab}\sigma^{ab} - \boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{\ell})$$

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Sketch of proof: start from the *Ricci identity* ($\equiv$ definition of the Riemann tensor $R^{\gamma}{}_{\delta\alpha\beta}$) applied to the vector field $\boldsymbol{\ell}$:

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)\,\ell^\gamma = R^{\gamma}{}_{\mu\alpha\beta}\,\ell^\mu$$

Contract over α and γ to make appear the Ricci tensor $\boldsymbol{R}$ and contract with ℓ^β:

$$\ell^\nu\nabla_\mu\nabla_\nu\ell^\mu - \ell^\nu\nabla_\nu\nabla_\mu\ell^\mu = R_{\mu\nu}\ell^\mu\ell^\nu$$

Use $\nabla_\nu\ell^\mu = \Theta^\mu{}_\nu - \omega_\nu\ell^\mu + \ell_\nu k^\sigma\nabla_\sigma\ell^\mu$ and $\nabla_\mu\ell^\mu = \theta_{(\boldsymbol{\ell})} + \kappa$, then $\Theta^\mu{}_\nu\ell^\nu = 0$, $\ell^\nu\nabla_\mu\Theta^\mu{}_\nu = -\Theta^\mu{}_\nu\nabla_\mu\ell^\nu = -\Theta_{\mu\nu}\Theta^{\mu\nu} = -\sigma_{ab}\sigma^{ab} - \dfrac{1}{n-2}\,\theta^2_{(\boldsymbol{\ell})}$, $\omega_\nu\ell^\nu = \kappa$ and $\ell^\nu\ell^\mu\nabla_\mu\omega_\nu = \ell^\mu\nabla_\mu\kappa - \kappa^2$ to get the result.

Evolution of the expansion

Null Raychaudhuri equation for general relativity

If the Einstein equation holds:

$$\boldsymbol{\nabla}_{\boldsymbol{\ell}}\, \theta_{(\boldsymbol{\ell})} = \kappa\theta_{(\boldsymbol{\ell})} - \frac{1}{n-2}\, \theta^2_{(\boldsymbol{\ell})} - \sigma_{ab}\sigma^{ab} - 8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell})$$

Outline

Distinguishing a black hole horizon from a generic null hypersurface

A naive definition of a black hole:

A **black hole** is a localized region of spacetime from which neither massive particles nor massless ones (photons) can escape.

- no-escape facet $\Longrightarrow$ boundary = null hypersurface
  But we don't want the interior of a future null cone in Minkowski spacetime to be called a black hole...

Distinguishing a black hole horizon from a generic null hypersurface

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- no-escape facet $\Longrightarrow$ boundary = null hypersurface
  But we don't want the interior of a future null cone in Minkowski spacetime to be called a black hole...
- localized facet: *for equilibrium configurations*, can be enforced by cross-sections = *compact* surfaces with *constant* area, i.e. vanishing expansion

Non-expanding horizons

Definition

A **non-expanding horizon (NEH)** is a null hypersurface $\mathscr{H}$ whose complete cross-sections $\mathscr{S}$ are *compact* manifolds (without boundary) and such that the expansion along any null normal $\boldsymbol{\ell}$ vanishes identically:

$$\boxed{\theta_{(\boldsymbol{\ell})} = 0}$$

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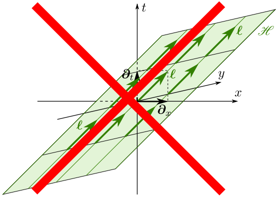
$$\boxed{\theta_{(\boldsymbol{\ell})} = 0}$$

Remark 1: definition independent of $\boldsymbol{\ell}$, due to $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell} \implies \theta_{(\boldsymbol{\ell}')} = \alpha\theta_{(\boldsymbol{\ell})}$
Remark 2: assuming that all cross-sections have the same topology, $\mathscr{H}$ has the "cylinder" topology: $\mathscr{H} \simeq \mathbb{R} \times \mathscr{S}$.

Remark 3: NEH concept introduced by P. Hájíček in 1973 [Com. Math. Phys. **34**, 37] under the name *perfect horizon*; the term *non-expanding horizon* has been coined by A. Ashtekar, S. Fairhurst & B. Krishnan in 2000 [PRD **62**, 104025].

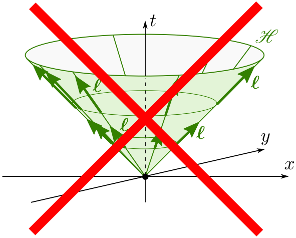
(Counter-)examples of non-expanding horizons

null hyperplane



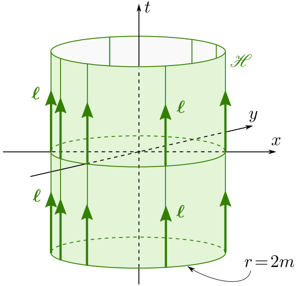
non-compact
cross-sections

future null cone



nonzero expansion

Schwarzschild horizon

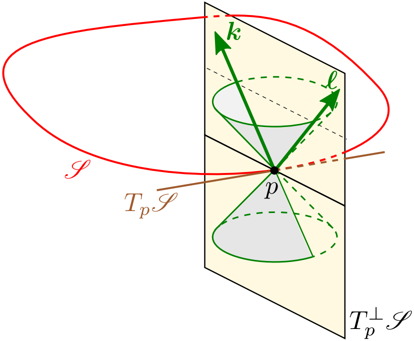


OK

Connection with marginally trapped surfaces

Definition of a trapped surface (1/2)

$\mathscr{S}$: closed (compact without boundary) spacelike $(n-2)$-dimensional surface embedded in spacetime $(\mathscr{M}, \boldsymbol{g})$



Being spacelike, $\mathscr{S}$ lies outside the light cone $\Longrightarrow \exists$ two future-directed null directions orthogonal to $\mathscr{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta_{(\boldsymbol{\ell})}$

$\boldsymbol{k}$ = ingoing, expansion $\theta_{(\boldsymbol{k})}$

In Minkowski spacetime:

$\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} > 0$

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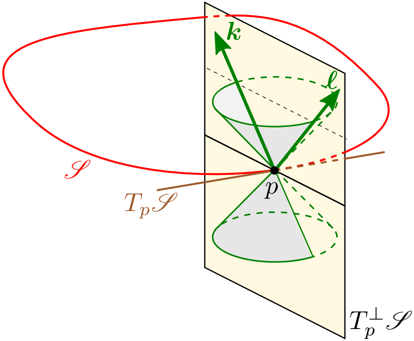

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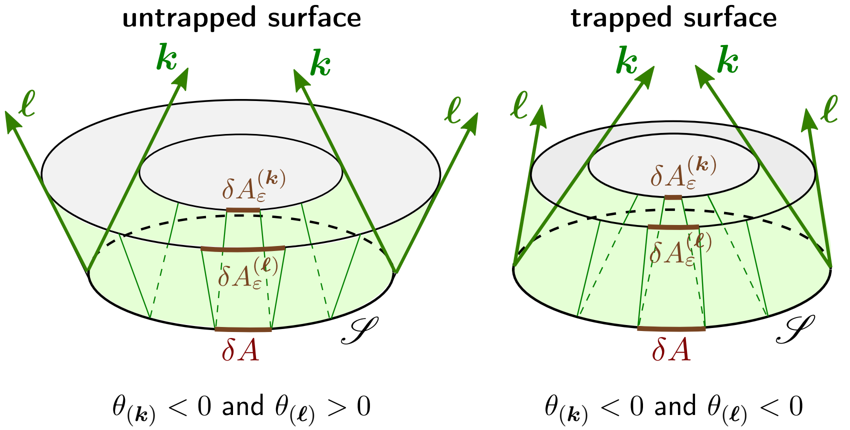
$\mathscr{S}$ is **trapped** $\Longleftrightarrow$ $\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} < 0$ [Penrose 1965]

$\mathscr{S}$ is **marginally trapped** $\Longleftrightarrow$ $\theta_{(\boldsymbol{k})} < 0$ and $\theta_{(\boldsymbol{\ell})} = 0$

Connection with marginally trapped surfaces

Definition of a trapped surface (2/2)

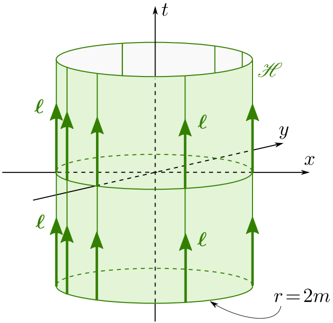


No trapped surface in Minkowski spacetime
$\Longrightarrow$ *trapped surface* = **local** concept characterizing strong gravity

Connection with marginally trapped surfaces

Generically, one has $\theta_{(\boldsymbol{k})} < 0$ along cross-sections of a non-expanding horizon. Hence:

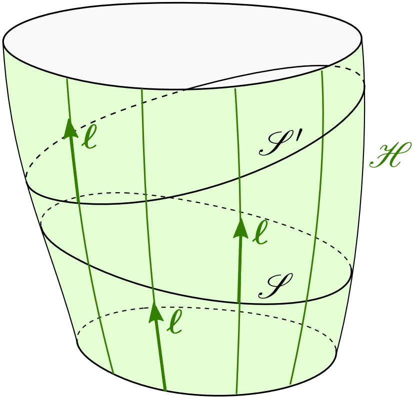
A non-expanding horizon is (generically) a null hypersurface foliated by marginally trapped surfaces.



Example: Schwarzschild horizon

$$\theta_{(\boldsymbol{k})} = -\frac{1}{m} \quad \text{and} \quad \theta_{(\boldsymbol{\ell})} = 0$$

Area of a non-expanding horizon



Each cross-section $\mathscr{S}$ of $\mathscr{H}$ is a *spacelike* closed surface.

The area of $\mathscr{S}$ is given by the positive definite metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ on $\mathscr{S}$:

$$A = \int_{\mathscr{S}} \sqrt{q}\, \mathrm{d}y^1 \cdots \mathrm{d}y^{n-2}$$

where $y^a = (y^1, \ldots, y^{n-2})$ are coordinates on $\mathscr{S}$ and $q := \det(q_{ab})$

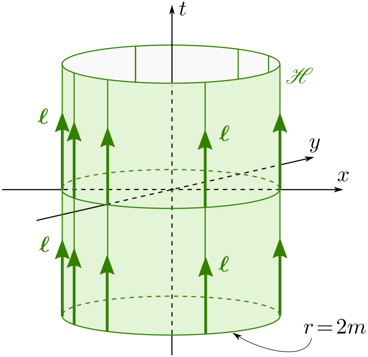
Since $\theta_{(\boldsymbol{\ell})} = 0$, we have:

On a non-expanding horizon, the area A is independent of the choice of the cross-section $\mathscr{S}$ $\Longrightarrow$ **area of $\mathscr{H}$**

Example: area of the Schwarzschild horizon

Spacetime metric:

$$\boldsymbol{g} = -\left(1 - \frac{2m}{r}\right)\mathbf{d}t^2 + \frac{4m}{r}\,\mathbf{d}t\,\mathbf{d}r + \left(1 + \frac{2m}{r}\right)\mathbf{d}r^2 + r^2\mathbf{d}\theta^2 + r^2\sin^2\theta\,\mathbf{d}\varphi^2$$



$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

Example: area of the Schwarzschild horizon

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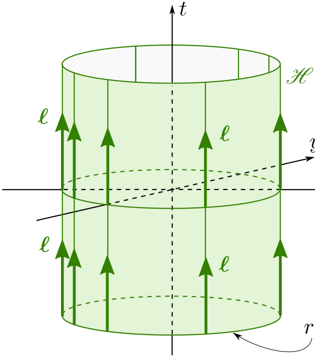

$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

$\Longrightarrow$ induced metric on $\mathscr{S}$:

$$q_{ab}\, \mathrm{d}y^a \mathrm{d}y^b = (2m)^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right)$$

$$\Longrightarrow q := \det(q_{ab}) = (2m)^4 \sin^2\theta$$

$$\Longrightarrow A = \int_{\mathscr{S}} (2m)^2 \sin\theta\, \mathrm{d}\theta \mathrm{d}\varphi$$

$$\Longrightarrow \boxed{A = 16\pi m^2}$$

Vanishing of the deformation rate tensor (1/2)

Null Raychaudhuri equation + $\theta_{(\boldsymbol{\ell})} = 0 \Longrightarrow \sigma_{ab}\sigma^{ab} + \boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{\ell}) = 0$

$\boldsymbol{q}$ Riemannian $\Longrightarrow \sigma_{ab}\sigma^{ab} = \sum_{a=1}^{n-2}\sum_{b=1}^{n-2}(\sigma_{ab})^2$ in a $\boldsymbol{q}$-orthonormal frame

$\Longrightarrow \sigma_{ab}\sigma^{ab} \geq 0$

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$\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0$ as well if one assumes the **null convergence condition**:

$$\boxed{\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0 \quad \text{for any null vector } \boldsymbol{\ell}}$$

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For general relativity, the null convergence condition is equivalent to the **null energy condition**:

$$\boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell}) \geq 0 \quad \text{for any null vector } \boldsymbol{\ell}$$

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NB: this is a very mild assumption, which is satisfied by

- vacuum ($\boldsymbol{T} = 0$), any electromagnetic field, any massless scalar field, dark energy ($\boldsymbol{T} = -(\Lambda/8\pi)\,\boldsymbol{g}$)
- any "standard" matter that fulfills the **weak energy condition**: $\boldsymbol{T}(\boldsymbol{u}, \boldsymbol{u}) \geq 0$ for any $\boldsymbol{u}$ timelike (positivity of the energy)

Vanishing of the deformation rate tensor (2/2)

If the null convergence condition is fulfilled, one has necessarily both $\boldsymbol{R}(\boldsymbol{\ell}, \boldsymbol{\ell}) = 0$ and $\sigma_{ab}\sigma^{ab} = 0$. Since $\boldsymbol{q}$ is Riemannian, the later implies $\boldsymbol{\sigma} = 0$. It follows that $\boldsymbol{\Theta} = \boldsymbol{\sigma} + \frac{1}{n-2}\theta_{(\boldsymbol{\ell})}\, \boldsymbol{q} = 0$. Hence

Invariance of the cross-section metric along the null generators of a NEH

Provided that the null convergence condition holds — which occurs in general relativity if the null energy condition holds —, the deformation rate of any cross-section $\mathscr{S}$ of a non-expanding horizon $\mathscr{H}$ along any null normal $\boldsymbol{\ell}$ is identically zero:

$$\boldsymbol{\Theta} := \frac{1}{2} \vec{\boldsymbol{q}}^{\,*} \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}}\, \boldsymbol{q} = 0$$

In other words, the whole metric $\boldsymbol{q}$ (and not only the area form, as a mere $\theta_{(\boldsymbol{\ell})} = 0$ would suggest) is invariant along the geodesic generators of $\mathscr{H}$.

Intrinsic affine connection on a NEH

Intrinsic affine connection ${}^{\mathscr{H}}\boldsymbol{\nabla}$

Let $\mathscr{H}$ be a NEH and $\mathfrak{X}(\mathscr{H})$ be the space of vector fields on $\mathscr{H}$. If the null convergence condition holds, the operator

$$\begin{array}{rcccl} {}^{\mathscr{H}}\boldsymbol{\nabla} & : & \mathfrak{X}(\mathscr{H}) \times \mathfrak{X}(\mathscr{H}) & \longrightarrow & \mathfrak{X}(\mathscr{H}) \\ & & (\boldsymbol{u}, \boldsymbol{v}) & \longmapsto & \boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{v}, \end{array}$$

is well-defined (i.e. ${}^{\mathscr{H}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ belongs to $\mathfrak{X}(\mathscr{H})$). Moreover, ${}^{\mathscr{H}}\boldsymbol{\nabla}$ fulfills all the properties of an affine connection, since $\boldsymbol{\nabla}$ does.

Proof:

$$\begin{aligned} \ell_\mu u^\nu \nabla_\nu v^\mu &= u^\nu \nabla_\nu (\underbrace{\ell_\mu v^\mu}_{0}) - v^\mu u^\nu \nabla_\nu \ell_\mu \\ &= -\underbrace{\Theta_{\nu\mu}}_{0} v^\mu u^\nu - \omega_\nu u^\nu \underbrace{\ell_\mu v^\mu}_{0} + v^\mu \underbrace{u^\nu \ell_\nu}_{0} k^\sigma \nabla_\sigma \ell_\mu = 0 \end{aligned}$$

Hence $\boldsymbol{\nabla}_{\boldsymbol{u}}\, \boldsymbol{v}$ is tangent to $\mathscr{H}$. □

A NEH is a totally geodesic null hypersurface

As a consequence of the identity ${}^{\mathscr{H}}\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v}$ for any pair $(\boldsymbol{u}, \boldsymbol{v})$ of vector fields tangent to $\mathscr{H}$:

$(\mathscr{H}, {}^{\mathscr{H}}\boldsymbol{\nabla})$ is a **totally geodesic submanifold** of $(\mathscr{M}, \boldsymbol{g})$, i.e. any geodesic of $(\mathscr{H}, {}^{\mathscr{H}}\boldsymbol{\nabla})$ is also a geodesic of $(\mathscr{M}, \boldsymbol{g})$.

Connection 1-form

Horizon-intrinsic derivative of the null normal to a NEH

If the null convergence condition holds, the derivative of the null normal $\boldsymbol{\ell}$ to a NEH $\mathscr{H}$ with respect to the intrinsic affine connection ${}^{\mathscr{H}}\!\boldsymbol{\nabla}$ takes the form

$$\boxed{{}^{\mathscr{H}}\!\boldsymbol{\nabla}\boldsymbol{\ell} = \boldsymbol{\ell} \otimes {}^{\mathscr{H}}\!\boldsymbol{\omega}}$$

where the **connection 1-form** ${}^{\mathscr{H}}\!\boldsymbol{\omega}$ is the tensor field on $\mathscr{H}$ that is the restriction to tangent vectors to $\mathscr{H}$ of the 1-form $\boldsymbol{\omega} := -\boldsymbol{k} \cdot \boldsymbol{\nabla}_{\boldsymbol{\Pi}}\boldsymbol{\ell}$. In other words, ${}^{\mathscr{H}}\!\boldsymbol{\omega} := \iota^*\boldsymbol{\omega}$ (pullback of $\boldsymbol{\omega}$ by the inclusion map $\iota : \mathscr{H} \to \mathscr{M}$).

Proof: ${}^{\mathscr{H}}\!\boldsymbol{\nabla}\boldsymbol{\ell}$ is a tensor field of type $(1,1)$ on $\mathscr{H}$, whose action on a pair (1-form $\boldsymbol{a}$ on $\mathscr{H}$, vector field $\boldsymbol{u}$ on $\mathscr{H}$) is ${}^{\mathscr{H}}\!\boldsymbol{\nabla}\boldsymbol{\ell}(\boldsymbol{a},\boldsymbol{u}) = \langle \boldsymbol{a}, {}^{\mathscr{H}}\!\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\ell}\rangle$. Introducing $\bar{\boldsymbol{a}} = \boldsymbol{a} \circ \boldsymbol{\Pi}$ as a 1-form on $\mathscr{M}$, we get $\langle \boldsymbol{a}, {}^{\mathscr{H}}\!\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\ell}\rangle = \langle \bar{\boldsymbol{a}}, {}^{\mathscr{H}}\!\boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\ell}\rangle$. Hence

$$
\begin{aligned}
{}^{\mathscr{H}}\!\boldsymbol{\nabla}\boldsymbol{\ell}(\boldsymbol{a},\boldsymbol{u}) &= \langle \bar{\boldsymbol{a}}, \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{\ell}\rangle = \bar{a}_\mu u^\nu \nabla_\nu \ell^\mu = \bar{a}_\mu u^\nu \Big(\underbrace{\Theta^\mu_{\ \ \nu}}_{0} + \omega_\nu \ell^\mu - \ell_\nu k^\rho \nabla_\rho \ell^\mu \Big) \\
&= \bar{a}_\mu \ell^\mu \, \omega_\nu u^\nu - \underbrace{\ell_\nu u^\nu}_{0} \bar{a}_\mu k^\rho \nabla_\rho \ell^\mu = \langle \bar{\boldsymbol{a}}, \boldsymbol{\ell}\rangle \langle \boldsymbol{\omega}, \boldsymbol{u}\rangle = \langle \boldsymbol{a}, \boldsymbol{\ell}\rangle \langle {}^{\mathscr{H}}\!\boldsymbol{\omega}, \boldsymbol{u}\rangle \qquad \square
\end{aligned}
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