Introduction to black hole physics

4. Dynamics of black holes

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Home page for the lectures

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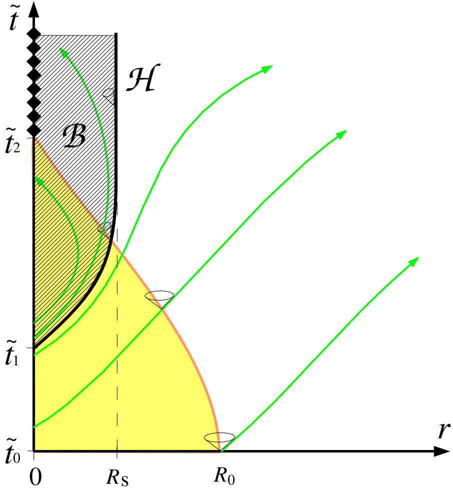
Lecture 4: Dynamics of black holes

Outline

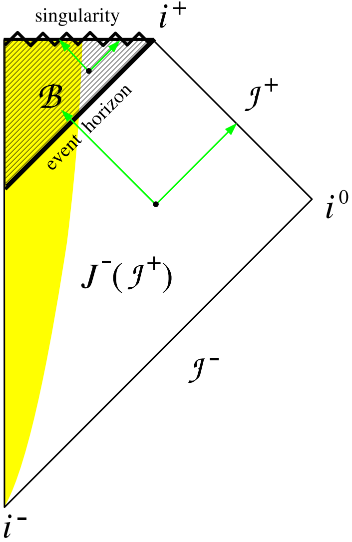
Gravitational collapse



Gravitational collapse of a star giving birth to black hole

yellow: matter; *orange:* stellar surface

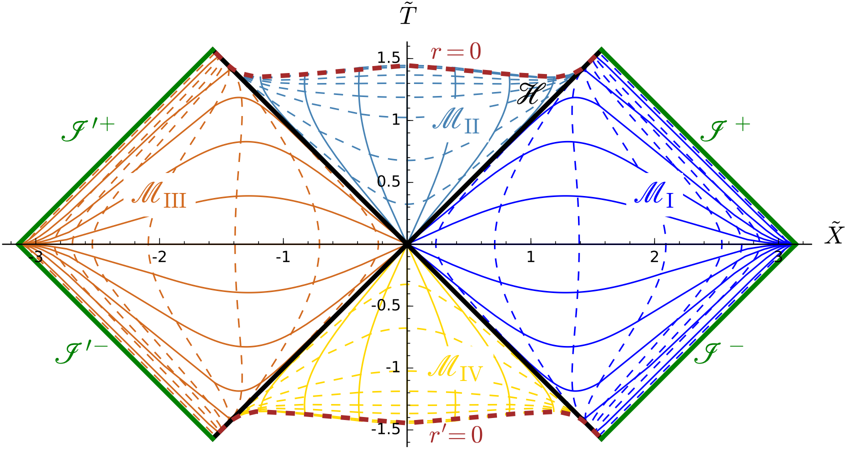
Gravitational collapse



Carter-Penrose conformal diagram

- $\mathscr{I}^+$: future null infinity
- $\mathscr{I}^-$: past null infinity
- $J^-(\mathscr{I}^+)$: causal past of $\mathscr{I}^+$
- $\mathscr{B} := \mathscr{M} \setminus (J^-(\mathscr{I}^+) \cap \mathscr{M})$, black hole region
- $\mathscr{H} = \partial\mathscr{B}$, event horizon

Compare with the "eternal" Schwarzschild black hole

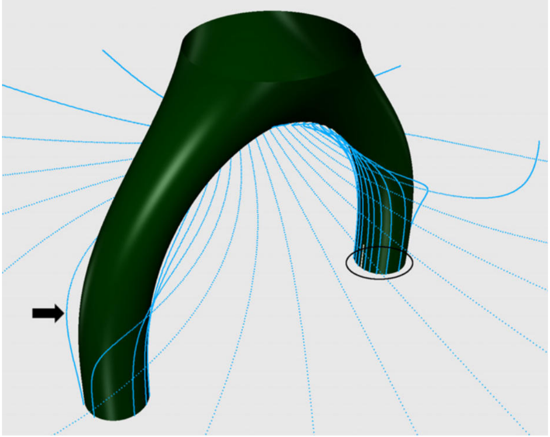


solid: $t = \text{const}$, *dashed:* $r = \text{const}$

(cf. Lecture 2)

Binary black hole merger

Head-on collision

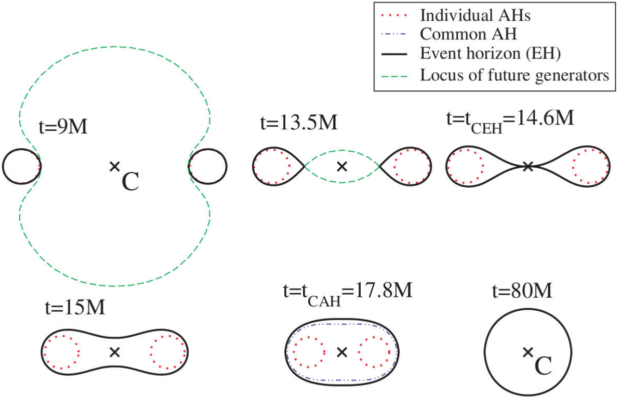


[Cohen, Pfeiffer & Scheel, CQG **26**, 035005 (2009)]

← event horizon of a head-on binary black hole merger (computed a posteriori)

blue curves: null geodesics that will eventually become become null generators of the event horizon.

Binary black hole merger
Head-on collision



[Cohen, Pfeiffer & Scheel, CQG **26**, 035005 (2009)]

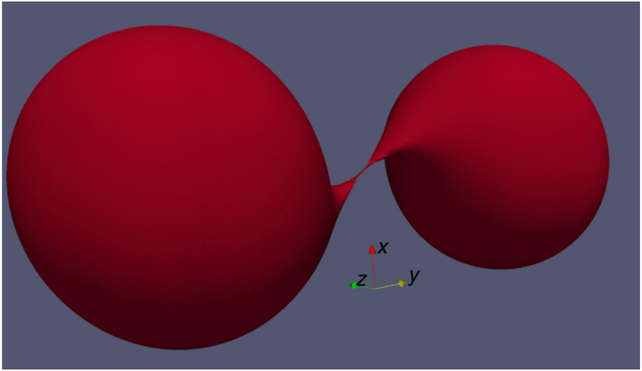
← cross-sections $\mathscr{S}_t$ of the event horizon $\mathscr{H}$ of a head-on binary black hole merger at various coordinate times t

black: $\mathscr{S}_t$

green dashed: trace of null geodesics that will become null generators of $\mathscr{H}$

red and blue dashed: apparent horizons (marginally trapped surfaces)

Binary black hole merger
Inspiral from circular orbit



[Cohen, Kaplan & Scheel, PRD **85**, 024031 (2012)]

← First connected cross-section of the event horizon of an inspiralling binary black hole merger (slicing by coordinate time t)

(x, y)-*axes:* orbital plane

Outline

Small change in a black hole equilibrium

Physical setup: Consider an initially isolated Kerr black hole, of parameters (m, a), that is perturbed by the arrival of some external body or some gravitational radiation. After some transitory dynamical regime (e.g. absorption of the external body and emission of gravitational waves), the black hole relaxes to a new equilibrium configuration. According to the no-hair theorem, the final state has to be a Kerr black hole, of parameters $(m + \delta m, a + \delta a)$ say.

Question: how do the black hole global properties evolve during the process?

Global properties of a Kerr black hole

As seen in Lecture 3, a Kerr black hole of parameters (m, a) has

- mass $M = m$
- angular momentum $J = am$
- area $A = 8\pi(M^2 + \sqrt{M^4 - J^2})$
- angular velocity $\Omega_H = \dfrac{J}{2M(M^2 + \sqrt{M^4 - J^2})}$
- surface gravity $\kappa = \dfrac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}$

Relating the change in M to that in A and J

From $A = 8\pi(M^2 + \sqrt{M^4 - J^2})$, we get:

$$\frac{1}{8\pi}\,\mathrm{d}A = 2M\,\mathrm{d}M + \frac{2M^3}{\sqrt{M^4 - J^2}}\,\mathrm{d}M - \frac{J}{\sqrt{M^4 - J^2}}\,\mathrm{d}J$$

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$$\Longrightarrow \frac{\sqrt{M^4 - J^2}}{8\pi}\,\mathrm{d}A = 2M(M^2 + \sqrt{M^4 - J^2})\,\mathrm{d}M - J\,\mathrm{d}J$$

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$$\Longrightarrow \mathrm{d}M = \frac{1}{8\pi}\underbrace{\frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}}_{\kappa}\,\mathrm{d}A + \underbrace{\frac{J}{2M(M^2 + \sqrt{M^4 - J^2})}}_{\Omega_H}\,\mathrm{d}J$$

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Hence

$$\mathrm{d}M = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega_H\,\mathrm{d}J$$

On the way to the first law of black hole dynamics...

$$\mathrm{d}M = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega_H\,\mathrm{d}J$$

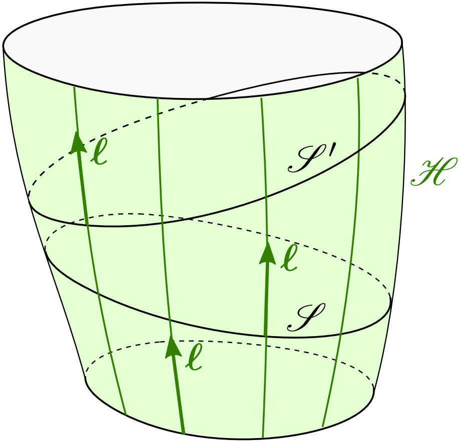
- $\mathrm{d}M$: change in energy
- $\Omega_H\,\mathrm{d}J$: change in rotational kinetic energy
- $\frac{\kappa}{8\pi}\,\mathrm{d}A$: ??

Looks premature to call this relation a first law of black hole (thermo)dynamics...

Outline

Evolution of cross-sections of an event horizon

Framework: generic (dynamical) black hole, event horizon $\mathscr{H}$



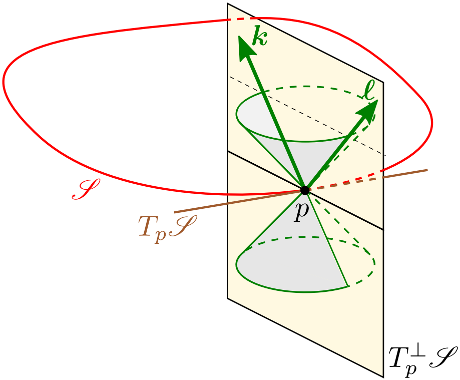
$\mathscr{H}$ is ruled by null geodesic generators (cf. Lecture 1).

Let $\boldsymbol{\ell}$ be the future-directed null normal vector field associated with a affine parameter λ of these geodesics:

$$\boldsymbol{\ell} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda} \quad \text{and} \quad \boldsymbol{\nabla}_{\boldsymbol{\ell}}\boldsymbol{\ell} = 0$$

Let us consider a cross-section $\mathscr{S}$ of $\mathscr{H}$ and study its *evolution along* $\boldsymbol{\ell}$ (Lie dragging of $\mathscr{S}_t$ along $\boldsymbol{\ell}$)

Evolution of cross-sections of an event horizon



$$T_p^\perp\mathscr{S} = \mathrm{Span}(\boldsymbol{k}, \boldsymbol{\ell})$$

Since $\mathscr{S}$ is a spacelike surface, for all $p \in \mathscr{S}$, the tangent space $T_p\mathscr{S}$ to $\mathscr{S}$ is a spacelike 2-plane and admits an orthogonal complement $T_p^\perp\mathscr{S}$, which is a timelike plane:

$$T_p\mathscr{M} = T_p\mathscr{S} \oplus T_p^\perp\mathscr{S}$$

The intersection of the null cone at p with $T_p^\perp\mathscr{S}$ define 2 null directions orthogonal to $\mathscr{S}$:

- one is along $\boldsymbol{\ell}$ (and thus tangent to $\mathscr{H}$)
- the other one is along a null vector $\boldsymbol{k}$, unambiguously defined via the normalization $\boldsymbol{g}(\boldsymbol{k}, \boldsymbol{\ell}) = -1$

Induced metric and orthogonal projector

We introduce the symmetric bilinear form $\boldsymbol{q}$ by

$$q_{\alpha\beta} = g_{\alpha\beta} + \ell_\alpha k_\beta + k_\alpha \ell_\beta$$

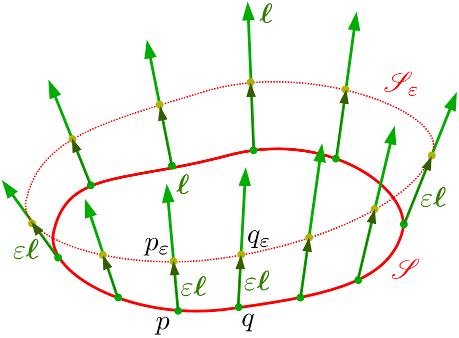
$\boldsymbol{q}$ is a spacetime extension of the metric induced by $\boldsymbol{g}$ on $\mathscr{S}$:
Proof: if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors tangent to $\mathscr{S}$:
$q_{\mu\nu}u^\mu v^\nu = g_{\mu\nu}u^\mu v^\nu + \underbrace{\ell_\mu u^\mu}_{0} k_\nu v^\nu + k_\mu u^\mu \underbrace{\ell_\nu v^\nu}_{0} = g_{\mu\nu}u^\mu v^\nu$,
i.e. $\boldsymbol{q}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{u},\boldsymbol{v})$. □

The orthogonal projector onto $\mathscr{S}$ is the type-(1,1) tensor $\vec{\boldsymbol{q}}$ whose components are deduced from those of $\boldsymbol{q}$ by raising the first index:

$$q^\alpha{}_\beta = \delta^\alpha{}_\beta + \ell^\alpha\, k_\beta + k^\alpha\, \ell_\beta$$

In particular $q^\alpha{}_\mu \ell^\mu = 0$ and $q^\alpha{}_\mu k^\mu = 0$.

Deformation rate tensor



The **deformation rate tensor of $\mathscr{S}$** measures the evolution of the metric $\boldsymbol{q}$ of $\mathscr{S}$ along the null normal $\boldsymbol{\ell}$, i.e. how the metric of $\mathscr{S}$ varies when $\mathscr{S}$ is Lie-dragged along $\boldsymbol{\ell}$. The relevant operator is then the Lie derivative along $\boldsymbol{\ell}$, $\mathcal{L}_{\boldsymbol{\ell}}$:

$$\boldsymbol{\Theta} := \frac{1}{2}\vec{\boldsymbol{q}}^*\mathcal{L}_{\boldsymbol{\ell}}\,\boldsymbol{q}$$

$$\iff \quad \Theta_{\alpha\beta} = \frac{1}{2}q^\mu{}_\alpha q^\nu{}_\beta \mathcal{L}_{\boldsymbol{\ell}}\, q_{\mu\nu}$$

Expressing the Lie derivative in terms of the covariant derivative, via $\mathcal{L}_{\boldsymbol{\ell}}\, q_{\mu\nu} = \ell^\sigma\nabla_\sigma q_{\mu\nu} + q_{\sigma\nu}\nabla_\mu\ell^\sigma + q_{\mu\sigma}\nabla_\nu\ell^\sigma$, we get

$$\Theta_{\alpha\beta} = q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \ell_\nu$$

Expansion and shear tensor

From Lecture 1, the **expansion along $\boldsymbol{\ell}$** is

$$\theta_{(\boldsymbol{\ell})} := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A}$$

Since in adapted coordinates $\delta A = \sqrt{q}\,\mathrm{d}y^1 \mathrm{d}y^2$, we get

$$\theta_{(\boldsymbol{\ell})} = \mathcal{L}_{\boldsymbol{\ell}} \ln\sqrt{q} = \frac{1}{2}\mathcal{L}_{\boldsymbol{\ell}} \ln q = \frac{1}{2} q^{ab} \mathcal{L}_{\boldsymbol{\ell}}\, q_{ab} = \frac{1}{2} q^{\mu\nu} \mathcal{L}_{\boldsymbol{\ell}}\, q_{\mu\nu} = g^{\mu\nu}\Theta_{\mu\nu}$$

Hence the expansion $\theta_{(\boldsymbol{\ell})}$ is nothing but the trace of the deformation rate tensor $\boldsymbol{\Theta}$.

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Hence the expansion $\theta_{(\boldsymbol{\ell})}$ is nothing but the trace of the deformation rate tensor $\boldsymbol{\Theta}$.

The trace-free part of $\boldsymbol{\Theta}$ defines the **shear tensor of $\mathscr{S}$**:

$$\boldsymbol{\sigma} := \boldsymbol{\Theta} - \frac{1}{2} \theta_{(\boldsymbol{\ell})}\, \boldsymbol{q} \quad \Longleftrightarrow \quad \sigma_{\alpha\beta} = \Theta_{\alpha\beta} - \frac{1}{2} \theta_{(\boldsymbol{\ell})}\, q_{\alpha\beta}$$

Null Raychaudhuri equation (1/3)

Starting point: Ricci identity ($\equiv$ definition of the Riemann tensor $R^{\gamma}{}_{\delta\alpha\beta}$) applied to the vector field $\boldsymbol{\ell}$:

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\, \ell^\gamma = R^{\gamma}{}_{\mu\alpha\beta}\, \ell^\mu$$

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Contracting over α and γ makes appear the Ricci tensor $R_{\alpha\beta} := R^{\sigma}{}_{\alpha\sigma\beta}$:

$$\nabla_\sigma \nabla_\beta \ell^\sigma - \nabla_\beta \nabla_\sigma \ell^\sigma = R_{\mu\beta} \ell^\mu$$

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Then, contract with $\boldsymbol{\ell}$ to get a scalar equation:

$$\ell^\nu \nabla_\mu \nabla_\nu \ell^\mu - \ell^\nu \nabla_\nu \nabla_\mu \ell^\mu = R_{\mu\nu} \ell^\mu \ell^\nu$$

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Using $\ell^\nu \nabla_\mu \nabla_\nu \ell^\mu = \nabla_\mu (\underbrace{\ell^\nu \nabla_\nu \ell^\mu}_{0}) - \nabla_\mu \ell^\nu \nabla_\nu \ell^\mu$ yields

$$\ell^\nu \nabla_\nu \nabla_\mu \ell^\mu = -\nabla_\mu \ell^\nu \nabla_\nu \ell^\mu - R_{\mu\nu} \ell^\mu \ell^\nu$$

Null Raychaudhuri equation (2/3)

Now, from $\Theta_{\alpha\beta} = q^{\mu}{}_{\alpha} q^{\nu}{}_{\beta} \nabla_{\mu} \ell_{\nu}$, with $q^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \ell^{\alpha} k_{\beta} + k^{\alpha} \ell_{\beta}$, we get

$$\nabla_{\alpha} \ell_{\beta} = \Theta_{\alpha\beta} - k^{\sigma} \nabla_{\alpha} \ell_{\sigma} \ell_{\beta} - k^{\rho} k^{\sigma} \nabla_{\rho} \ell_{\sigma} \ell_{\alpha} \ell_{\beta} - \ell_{\alpha} k^{\sigma} \nabla_{\sigma} \ell_{\beta}$$

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$$\nabla_{\alpha} \ell_{\beta} = \Theta_{\alpha\beta} - k^{\sigma} \nabla_{\alpha} \ell_{\sigma} \, \ell_{\beta} - k^{\rho} k^{\sigma} \nabla_{\rho} \ell_{\sigma} \, \ell_{\alpha} \ell_{\beta} - \ell_{\alpha} k^{\sigma} \nabla_{\sigma} \ell_{\beta}$$

from which (using $\Theta_{\alpha\mu} \ell^{\mu} = 0$, $\ell_{\mu} \ell^{\mu} = 0$, $\ell_{\mu} \nabla_{\alpha} \ell^{\mu} = 0$ and $\ell^{\mu} \nabla_{\mu} \ell^{\alpha} = 0$),

$$\nabla_{\mu} \ell^{\mu} = \Theta^{\mu}{}_{\mu} = \theta_{(\boldsymbol{\ell})}$$

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$$\nabla_{\mu}\ell^{\mu} = \Theta^{\mu}{}_{\mu} = \theta_{(\boldsymbol{\ell})}$$

and

$$\begin{aligned} \nabla_{\mu}\ell^{\nu}\nabla_{\nu}\ell^{\mu} &= \Theta_{\mu\nu}\Theta^{\mu\nu} = \Theta_{ab}\Theta^{ab} \\ &= \left(\sigma_{ab} + \tfrac{1}{2}\theta_{(\boldsymbol{\ell})}\, q_{ab}\right)\left(\sigma^{ab} + \tfrac{1}{2}\theta_{(\boldsymbol{\ell})}\, q^{ab}\right) \\ &= \sigma_{ab}\sigma^{ab} + \tfrac{1}{4}(\theta_{(\boldsymbol{\ell})})^2 \underbrace{q_{ab}q^{ab}}_{2} \\ &= \sigma_{ab}\sigma^{ab} + \tfrac{1}{2}(\theta_{(\boldsymbol{\ell})})^2 \end{aligned}$$

Null Raychaudhuri equation (3/3)

Hence

$$\ell^\mu \nabla_\mu \theta_{(\boldsymbol{\ell})} = -\frac{1}{2}(\theta_{(\boldsymbol{\ell})})^2 - \sigma_{ab}\sigma^{ab} - R_{\mu\nu}\ell^\mu \ell^\nu$$

Null Raychaudhuri equation (3/3)

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$$\ell^\mu \nabla_\mu \theta_{(\boldsymbol{\ell})} = -\frac{1}{2}(\theta_{(\boldsymbol{\ell})})^2 - \sigma_{ab}\sigma^{ab} - R_{\mu\nu}\ell^\mu \ell^\nu$$

Finally one invokes Einstein equation:

$$R_{\mu\nu}\ell^\mu \ell^\nu - \frac{1}{2}R\,\underbrace{g_{\mu\nu}\ell^\mu \ell^\nu}_{0} = 8\pi T_{\mu\nu}\ell^\mu \ell^\nu$$

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and use $\boldsymbol{\ell} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda}$ to rewrite $\ell^\mu \nabla_\mu \theta_{(\boldsymbol{\ell})} = \frac{\mathrm{d}\theta_{(\boldsymbol{\ell})}}{\mathrm{d}\lambda}$ and get

Null Raychaudhuri equation

$$\frac{\mathrm{d}\theta_{(\boldsymbol{\ell})}}{\mathrm{d}\lambda} = -\frac{1}{2}(\theta_{(\boldsymbol{\ell})})^2 - \sigma_{ab}\sigma^{ab} - 8\pi T_{\mu\nu}\ell^\mu \ell^\nu$$

Evolution of the expansion along a null geodesic generator of $\mathscr{H}$

Null energy condition

Physical assumption: in the vicinity of the event horizon $\mathscr{H}$, matter and (non-gravitational) fields obey the null energy condition:

$$\boxed{T_{\mu\nu}\ell^\mu\ell^\nu \geq 0} \text{ for any null vector } \boldsymbol{\ell}$$

NB: this is a very mild assumption:

- it is trivially satisfied by vacuum: $T_{\mu\nu} = 0$
- it is satisfied by any electromagnetic field $\boldsymbol{F}$:

  $$T_{\mu\nu}\ell^\mu\ell^\nu = \frac{1}{\mu_0}\Big(F_{\sigma\mu}\ell^\mu F^\sigma{}_\nu\ell^\nu - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\underbrace{g_{\mu\nu}\ell^\mu\ell^\nu}_{0}\Big) = \frac{1}{\mu_0}E_\mu E^\mu,$$

  with $E^\alpha := F^\alpha{}_\mu\ell^\mu$ being necessarily spacelike or colinear to $\boldsymbol{\ell}$, since it is orthogonal to $\boldsymbol{\ell}$ thanks to the antisymmetry of $\boldsymbol{F}$: $E_\mu\ell^\mu = F_{\mu\nu}\ell^\nu\ell^\mu = 0$; hence $E_\mu E^\mu \geq 0$ and $T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$
- it is implied by the weak energy condition, which shall be obeyed by any "reasonable" matter: $T_{\mu\nu}u^\mu u^\nu \geq 0$ for any $\boldsymbol{u}$ timelike (positivity of the energy)

Positivity of the expansion on $\mathscr{H}$

Positive expansion theorem

On a black hole event horizon $\mathscr{H}$, the expansion along any future-directed null normal $\boldsymbol{\ell}$ is everywhere positive or zero:

$$\theta_{(\boldsymbol{\ell})} \geq 0$$

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Proof: if $\boldsymbol{\ell}$ is only pregeodesic ($\kappa \neq 0$), rescale it $\tilde{\boldsymbol{\ell}} = \alpha \boldsymbol{\ell}$, $\alpha > 0$ to get a geodesic vector field ($\boldsymbol{\nabla}_{\tilde{\boldsymbol{\ell}}} \tilde{\boldsymbol{\ell}} = 0$), then $\theta_{(\tilde{\boldsymbol{\ell}})} = \alpha \theta_{(\boldsymbol{\ell})} \geq 0 \iff \theta_{(\boldsymbol{\ell})} \geq 0$.

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Consider the null Raychaudhuri equation along with the null energy condition:

$$\frac{\mathrm{d}\theta_{(\boldsymbol{\ell})}}{\mathrm{d}\lambda} = -\frac{1}{2}(\theta_{(\boldsymbol{\ell})})^2 \underbrace{-\sigma_{ab}\sigma^{ab}}_{\leq 0} \underbrace{-8\pi T_{\mu\nu}\ell^\mu \ell^\nu}_{\leq 0}$$

where $-\sigma_{ab}\sigma^{ab} \leq 0$ follows from the fact that σ_{ab} is a symmetric matrix in the 2-dimensional vector space $T_p\mathscr{S}$, equipped with the positive definite metric $\boldsymbol{q}$. It can thus be diagonalized, so that, in a $\boldsymbol{q}$-orthonormal basis, $\sigma_{ab} = \mathrm{diag}(\sigma, -\sigma)$, with $\sigma \in \mathbb{R}$, so that $\sigma_{ab}\sigma^{ab} = 2\sigma^2 \geq 0$.

Positivity of the expansion on $\mathscr{H}$

We have then necessarily $\dfrac{\mathrm{d}\theta_{(\boldsymbol{\ell})}}{\mathrm{d}\lambda} \leq -\dfrac{1}{2}(\theta_{(\boldsymbol{\ell})})^2$

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Assume that $\theta_{(\boldsymbol{\ell})} = \theta_0 < 0$ at some point $p \in \mathscr{H}$. Let us choose the affine parameter λ of the null geodesic generator $\mathscr{L}$ through p such that $\lambda = 0$ at p. The above equation implies

$$\forall \lambda \geq 0, \quad \theta_{(\boldsymbol{\ell})}(\lambda) \leq \bar{\theta}(\lambda)$$

where $\bar{\theta}(\lambda)$ obeys $\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\lambda} = -\frac{1}{2}\bar{\theta}^2$ with $\bar{\theta}(0) = \theta_0 \quad \Longrightarrow \quad \bar{\theta}(\lambda) = \frac{\theta_0}{1 + \theta_0 \lambda / 2}$

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$$\forall \lambda \geq 0, \quad \theta_{(\boldsymbol{\ell})}(\lambda) \leq \bar{\theta}(\lambda)$$

where $\bar{\theta}(\lambda)$ obeys $\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\lambda} = -\frac{1}{2}\bar{\theta}^2$ with $\bar{\theta}(0) = \theta_0 \quad \Longrightarrow \quad \bar{\theta}(\lambda) = \frac{\theta_0}{1 + \theta_0\lambda/2}$

$\Longrightarrow \bar{\theta} \to -\infty$ as $\lambda \to -2/\theta_0 > 0$

$\Longrightarrow \theta_{(\boldsymbol{\ell})} \to -\infty$ as $\lambda \to \lambda_*$ with $0 < \lambda_* \leq -2/\theta_0$

Positivity of the expansion on $\mathscr{H}$

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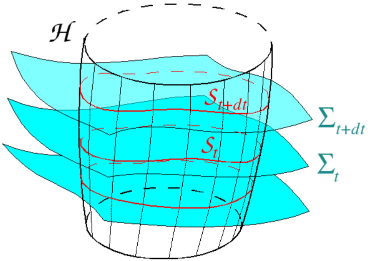
$\Longrightarrow$ $\theta_{(\boldsymbol{\ell})} \to -\infty$ as $\lambda \to \lambda_*$ with $0 < \lambda_* \leq -2/\theta_0$

$\Longrightarrow$ the point $p_* \in \mathscr{L}$ of parameter λ_* is a focusing point, i.e. a point where neighbouring null geodesic generators of $\mathscr{H}$ intersect

$\Longrightarrow$ contradiction with Penrose theorem (see Property 3 of event horizons in Lecture 1): there must be exactly one null geodesic generator through each point of $\mathscr{H}$, except at points where null geodesic generators enter $\mathscr{H}$ (p_* cannot be such point since $\lambda_* > 0$).

Second law of black hole dynamics



Second law (Hawking 1971)

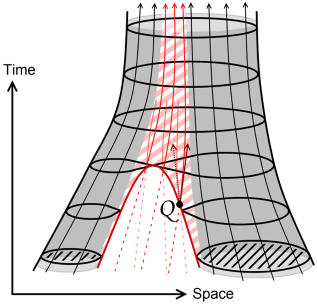
Let $(\mathscr{M}, \boldsymbol{g})$ be a spacetime containing a black hole of event horizon $\mathscr{H}$. Assume $\exists$ an open region $\mathscr{V} \supset \overline{\mathscr{M} \cap J^-(\mathscr{I}^+)}$ that is globally hyperbolic (no naked singularity, no Cauchy horizon).

Consider a foliation of $\mathscr{V}$ by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$, with t increasing towards the future, such that each Σ_t is a Cauchy hypersurface for $\mathscr{V}$.

Let $A(t)$ be the area of the cross-section $\mathscr{S}_t = \mathscr{H} \cap \Sigma_t$. Then, assuming the Einstein equation and the null energy condition,

$$\boxed{\frac{\mathrm{d}A}{\mathrm{d}t} \geq 0}$$

Second law of black hole dynamics



[Hamerly & Chen, PRD **84**, 124015 (2011)]

Proof: Let $\boldsymbol{\ell}$ be the null normal of $\mathscr{H}$ compatible with the foliation $(\mathscr{S}_t)_{t\in\mathbb{R}}$, i.e. such that $\boldsymbol{\nabla}_{\boldsymbol{\ell}}\, t = 1$. If there is no null geodesic entering $\mathscr{H}$ between $\mathscr{S}_t$ and $\mathscr{S}_{t+\mathrm{d}t}$, the cross-section $\mathscr{S}_{t+\mathrm{d}t}$ is deduced from $\mathscr{S}_t$ by Lie dragging along $\boldsymbol{\ell}$ by the parameter $\mathrm{d}t$ (see Lecture 1). By the very definition of $\theta_{(\boldsymbol{\ell})}$, we have then

$$\frac{\mathrm{d}A}{\mathrm{d}t} \geq \int_{\mathscr{S}_t} \theta_{(\boldsymbol{\ell})} \sqrt{q}\, \mathrm{d}y^1 \mathrm{d}y^2$$

with equality iff no new null geodesic is entering $\mathscr{H}$ (as the ones depicted in orange in the adjacent figure)

If Einstein equation and the null energy condition hold, then the result follows from the positive expansion theorem: $\theta_{(\boldsymbol{\ell})} \geq 0$. □

Outline

The first law revisited

$$\mathrm{d}M = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega_H\,\mathrm{d}J$$

Second law $\Longrightarrow$ A can only increase towards the future
$\Longrightarrow$ A may be identified with some entropy and κ with some temperature, to get a $T\mathrm{d}S$ term in the first law:

$$S = \alpha A$$

$$T = \frac{1}{8\pi\alpha}\kappa$$

with α to be determined...

Zeroth law

For a Kerr black hole: $\kappa = \dfrac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$

$\Longrightarrow$ κ is constant (i.e. it does not depend on θ).

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$\Longrightarrow$ κ is constant (i.e. it does not depend on θ).

More generally, one can show

Zeroth law of black hole mechanics

Let $\mathscr{H}$ be a Killing horizon and κ the non-affinity coefficient of the null normal coinciding with the Killing vector field on $\mathscr{H}$. If the matter and the non-gravitational fields obey the null dominant energy condition on $\mathscr{H}$, κ is uniform over $\mathscr{H}$:

$$\kappa = \mathrm{const}$$

Null dominant energy condition: $-T^{\alpha}{}_{\mu}\ell^{\mu}$ is future-directed null or timelike for any future-directed null vector $\boldsymbol{\ell}$

NB: null dominant energy condition $\Longrightarrow$ null energy condition used for the second law

Third law

For a Kerr black hole: $\kappa = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$

Hence $\kappa = 0 \iff a = m$ (extremal Kerr black hole)

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Third law

The surface gravity κ of a black hole in equilibrium cannot be reduced to zero by a finite sequence of operations.

The four laws of black hole thermodynamics

Zeroth law

The surface gravity κ of a black hole in equilibrium is constant

First law

Two nearby black hole equilibrium configurations are related by

$$\mathrm{d}M = \frac{\kappa}{8\pi}\,\mathrm{d}A + \Omega_H\,\mathrm{d}J$$

Second law

The area A of cross-sections of a black hole event horizon can only increase towards the future:

$$\frac{\mathrm{d}A}{\mathrm{d}t} \geq 0$$

Third law

The surface gravity κ of a black hole in equilibrium cannot be reduced to zero by a finite sequence of operations.

Set of four laws first formulated in Les Houches!

The Four Laws of Black Hole Mechanics

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[Commun. Math. Phys., **31**, 161 (1973)]

Hawking radiation enters the game...

Hawking radiation:

black-body radiation at $\boxed{T = \frac{\hbar}{2\pi k}\,\kappa}$ **(Hawking temperature)**

with k = Boltzmann constant

$\frac{\kappa}{8\pi}\mathrm{d}A = T\mathrm{d}S \Longrightarrow \boxed{S = \frac{k}{4}\frac{A}{\ell_{\mathrm{P}}^2}}$ **(Bekenstein-Hawking entropy)**

with $\ell_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}}$ = Planck length $\simeq 1.6\ 10^{-35}\ \mathrm{m}$

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For a Schwarzschild black hole of mass M: $\kappa = (4M)^{-1}$ and $A = 16\pi M^2$

$\Longrightarrow T = 6\ 10^{-8}\left(\frac{M_\odot}{M}\right)\ \mathrm{K}$ and $S = 1.1\ 10^{77}\left(\frac{M}{M_\odot}\right)^2 k$!!!

Outline

Upper bound on energy extracted via Penrose process

Consider some Penrose process (cf. Lecture 3) extracting energy from a Kerr black hole of initial mass $m_{\rm i}$ and specific angular momentum $a_{\rm i}$, the extraction taking place until the black hole angular momentum has decayed to zero (no longer any ergoregion outside the event horizon).
The final state is then a Schwarzschild black hole of mass $m_{\rm f}$ and the total amount of extracted energy is

$$\Delta E = m_{\rm i} - m_{\rm f}$$

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The final state is then a Schwarzschild black hole of mass $m_{\rm f}$ and the total amount of extracted energy is

$$\Delta E = m_{\rm i} - m_{\rm f}$$

Second law $\Longrightarrow A_{\rm f} \geq A_{\rm i}$, i.e. $2m_{\rm f}^2 \geq m_{\rm i}\left(m_{\rm i} + \sqrt{m_{\rm i}^2 - a_{\rm i}^2}\right)$

ΔE is maximal if $m_{\rm f}$ is minimal; given the above constraint, this is achieved for $\boxed{a_{\rm i} = m_{\rm i}} \Longrightarrow 2m_{\rm f}^2 \geq m_{\rm i}^2 \Longrightarrow m_{\rm f} \geq 2^{-1/2} m_{\rm i}$

$$\Longrightarrow \boxed{\Delta E \leq \left(1 - 2^{-1/2}\right) m_{\rm i} \simeq 0.29\, m_{\rm i}}$$

Upper bound on gravitational radiation from a BH merger
(Hawking, 1971)

Consider a binary black hole merger:

- initial stage: two far apart Kerr BH: (m_1, a_1) and (m_2, a_2)
- final stage: a single Kerr BH: (m_3, a_3)

The total amount of energy radiated as gravitational waves is
$\Delta E = m_1 + m_2 - m_3$
$\Longrightarrow$ efficiency of gravitational radiation: $\epsilon = \dfrac{m_1 + m_2 - m_3}{m_1 + m_2}$

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Second law $\Longrightarrow$ $A_3 \geq A_1 + A_2$, i.e.

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq m_1 \left(m_1 + \sqrt{m_1^2 - a_1^2} \right) + m_2 \left(m_2 + \sqrt{m_2^2 - a_2^2} \right)$$

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ϵ is maximal if m_3 is minimal; given the above constraint, this is achieved for $a_1 = m_1$, $a_2 = m_2$ and $a_3 = 0$

$\implies 2m_3^2 \geq m_1^2 + m_2^2 \implies m_3 \geq \sqrt{(m_1^2+m_2^2)/2}$

$\implies \epsilon \leq 1 - \dfrac{\sqrt{m_1^2+m_2^2}}{\sqrt{2}(m_1+m_2)}$

The maximum of the r.h.s. is achieved for $m_1 = m_2$ and is $1/2$, hence the upper bound:

$$\boxed{\epsilon \leq \frac{1}{2}}$$

Upper bound on gravitational radiation from a BH merger

Case of initially non-spinning equal-mass BH (Hawking, 1971)

Initially non-spinning equal-mass BH: $a_1 = a_2 = 0$ and $m_1 = m_2$
The second law constraint becomes

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq 4m_1^2$$

Again, ϵ is maximal if m_3 is minimal; given the above constraint, this is achieved for $a_3 = 0 \implies 2m_3^2 \geq 4m_1^2 \implies m_3 \geq \sqrt{2}m_1$
Hence the upper bound:

$$\boxed{\epsilon \leq 1 - 2^{-1/2} \simeq 0.29}$$

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Inspiralling binary BH merger with $m_1 = m_2$ and $a_1 = a_2 = 0$:
numerical relativity $\Longrightarrow a_3 = 0.68$ and $\epsilon = 0.048$
[Scheel et al., PRD **79**, 024003 (2009)]