

Figures of lecture 2

Geometry of null hypersurfaces and Killing horizons

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<https://relativite.obspm.fr/blackholes/paris25/>

PSL graduate program in Physics / Doctoral School PIF
ENS, Paris, France
7 May 2025

Home page for the lectures

<https://relativite.obspm.fr/blackholes/paris25/>

includes

- these slides
- the lecture notes (draft)
- some SageMath notebooks

Prerequisite

An introductory course on general relativity

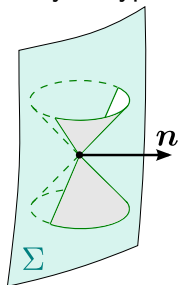
Hypersurfaces in spacetime

A **hypersurface** of the n -dimensional spacetime $(\mathcal{M}, g)$ is an embedded submanifold of $\mathcal{M}$ of dimension $n - 1$ (codimension 1).

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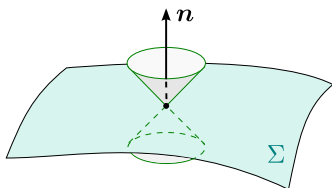
Locally, a hypersurface Σ can be of one of 3 types (n = normal to Σ):



Σ timelike

$g|_{\Sigma}$ Lorentzian

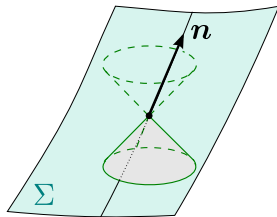
n spacelike



Σ spacelike

$g|_{\Sigma}$ Riemannian

n timelike

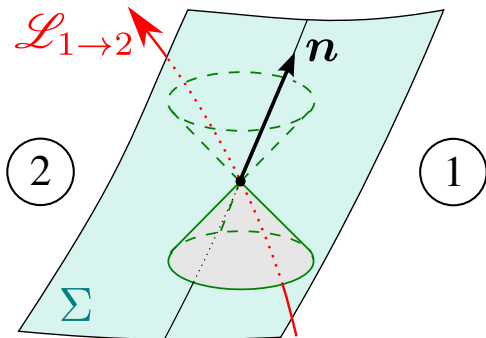


Σ null

$g|_{\Sigma}$ degenerate

n null (and tangent to Σ)

Null hypersurface as a causal boundary

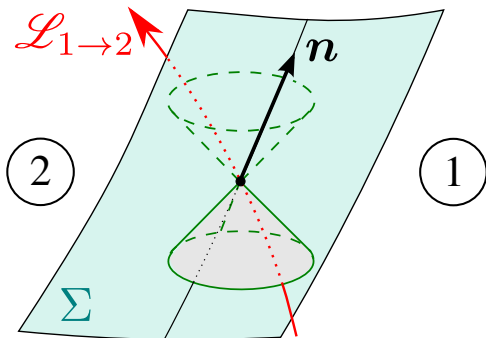


For timelike worldlines $\mathcal{L}$
directed towards the future:

null hypersurface = 1-way
membrane

$\Rightarrow$ eligible for a black hole
boundary...

Null hypersurface as a causal boundary



For timelike worldlines $\mathcal{L}$ directed towards the future:

null hypersurface = 1-way
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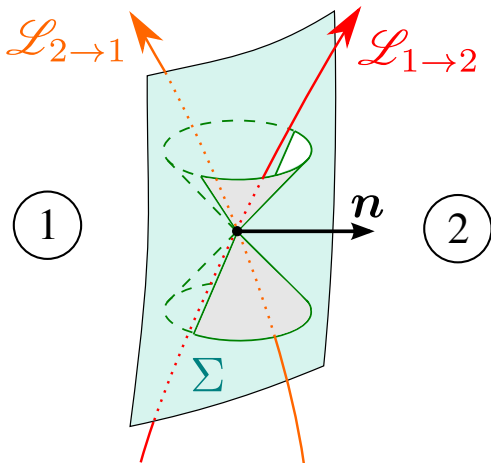
$\Rightarrow$ eligible for a black hole
boundary...

...and elected! (as a
consequence of the formal
definition of a black hole)

Theorem (Penrose 1968)

Wherever it is smooth, the event horizon of a black hole is a null hypersurface.

Timelike hypersurfaces are not causal boundaries

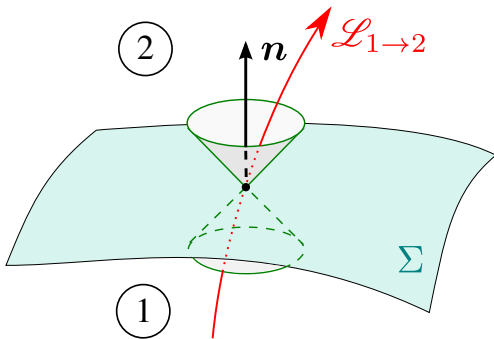


For timelike worldlines $\mathcal{L}$
directed towards the future:

timelike hypersurface = 2-way
membrane

$\Rightarrow$ not eligible for a black
hole boundary

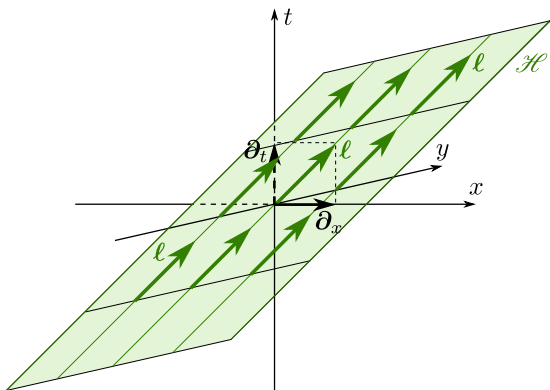
Spacelike hypersurfaces



For timelike worldlines $\mathcal{L}$
directed towards the future:

spacelike hypersurface =
1-way membrane
 $\Rightarrow$ in the dynamical black
hole context: trapping
horizons = spacelike
hypersurfaces

Example 1: null hyperplane in Minkowski spacetime



$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

$$u := t - x = 0$$

$$du = dt - dx$$

$$(du)_\alpha = \nabla_\alpha u = (1, -1, 0, 0)$$

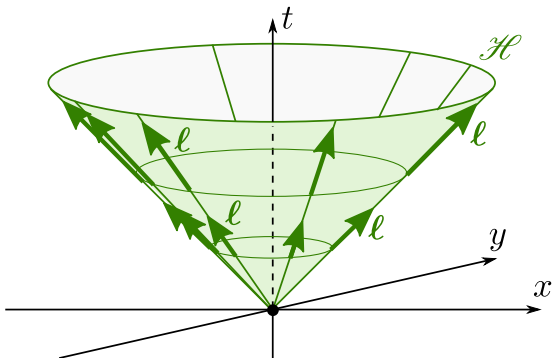
$$\nabla^\alpha u = (-1, -1, 0, 0)$$

Choose $\rho = 0$

$$\Rightarrow \ell^\alpha = (1, 1, 0, 0)$$

$$\ell = \partial_t + \partial_x$$

Example 2: future null cone in Minkowski spacetime



$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

$$u := t - \sqrt{x^2 + y^2 + z^2} = 0$$

$$du = dt - \frac{x}{r}dx - \frac{y}{r}dy - \frac{z}{r}dz$$

$$r := \sqrt{x^2 + y^2 + z^2}$$

$$\nabla_\alpha u = \left(1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

$$\nabla^\alpha u = \left(-1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

Choose $\rho = 0$

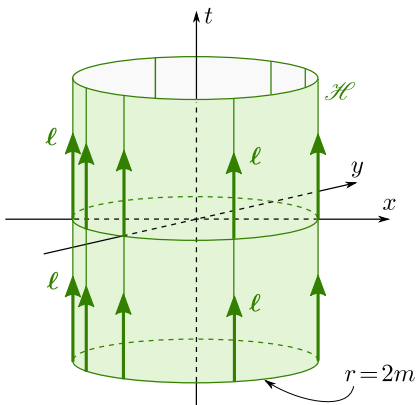
$$\Rightarrow \ell^\alpha = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$\ell = \partial_t + \frac{x}{r}\partial_x + \frac{y}{r}\partial_y + \frac{z}{r}\partial_z$$

Example 3: Schwarzschild horizon

in Eddington-Finkelstein coordinates

$$g = - \left(1 - \frac{2m}{r}\right) dt^2 + \frac{4m}{r} dt dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$



$$u := \left(1 - \frac{r}{2m}\right) \exp\left(\frac{r-t}{4m}\right) = 0$$

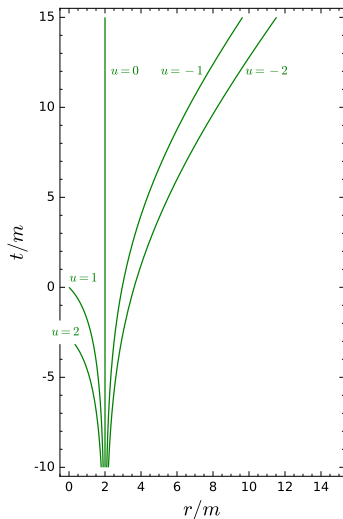
$$\mathcal{H} : u = 0 \iff r = 2m$$

$$du = \frac{1}{4m} e^{(r-t)/(4m)} \left[- \left(1 - \frac{r}{2m}\right) dt - \left(1 + \frac{r}{2m}\right) dr \right]$$

Exercise: compute ℓ with ρ chosen so that $\ell^t = 1$ and get

$$\ell = \partial_t + \frac{r-2m}{r+2m} \partial_r \implies \ell^{\mathcal{H}} = \partial_t$$

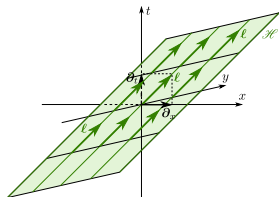
Example 3: Schwarzschild horizon in Eddington-Finkelstein coordinates



Hypersurfaces of constant value of u
around the Schwarzschild horizon $u = 0$

Examples of null geodesic generators

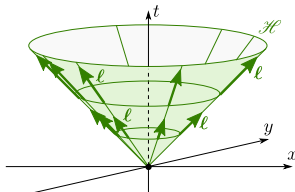
null hyperplane



$$\nabla_{\ell} \ell = 0$$

$$\kappa = 0$$

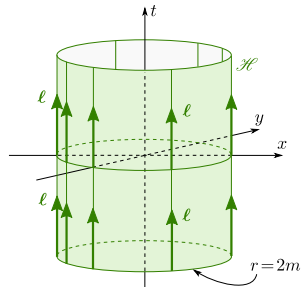
future null cone



$$\nabla_{\ell} \ell = 0$$

$$\kappa = 0$$

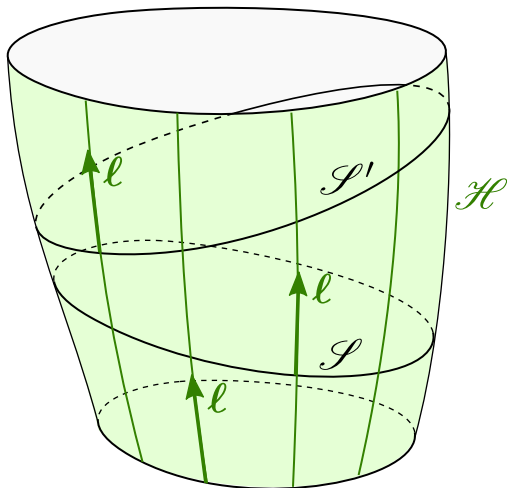
Schwarzschild horizon



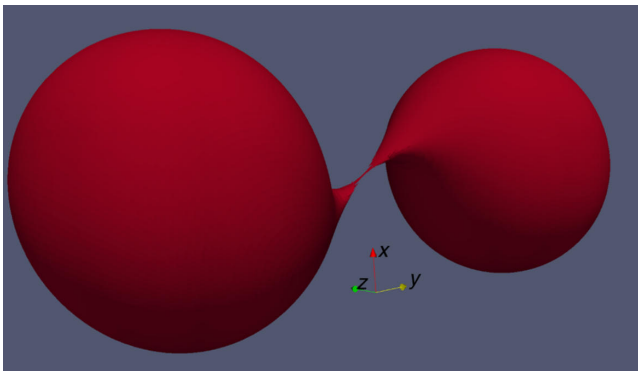
$$\nabla_{\ell} \ell = \kappa \ell$$

$$\kappa = \frac{1}{4m}$$

Cross-sections of a null hypersurface



Cross-section of the event horizon of a binary black hole merger

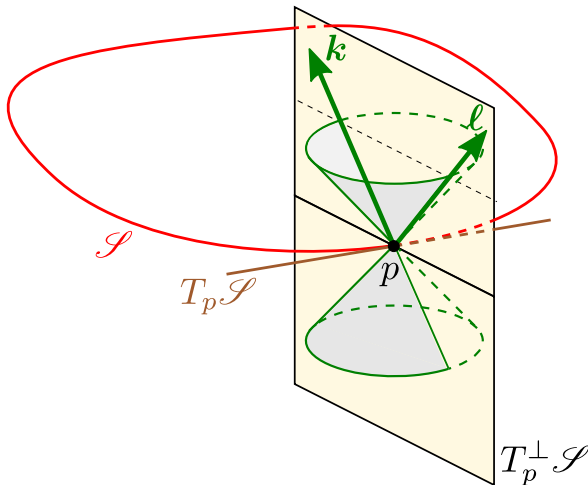


← First connected **cross-section** of the event horizon of an inspiralling binary black hole merger (slicing by coordinate time t)

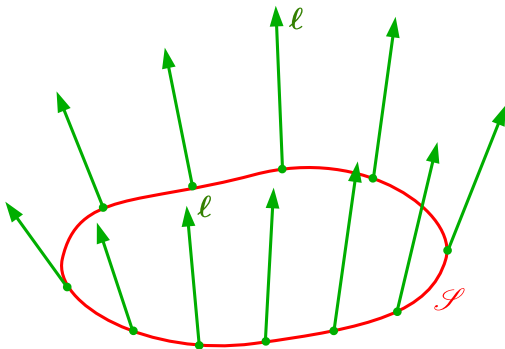
(x, y) -axes: orbital plane

[Cohen, Kaplan & Scheel, PRD **85**, 024031 (2012)]

Orthogonal complement of a cross-section tangent space

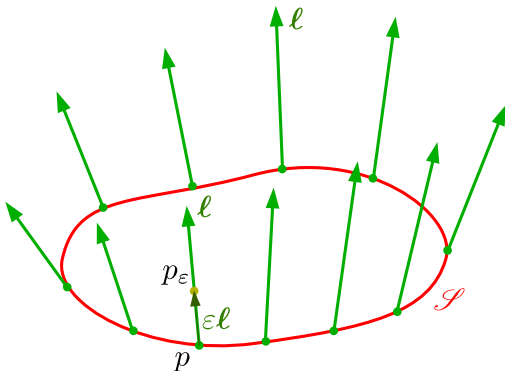


Expansion along a null normal



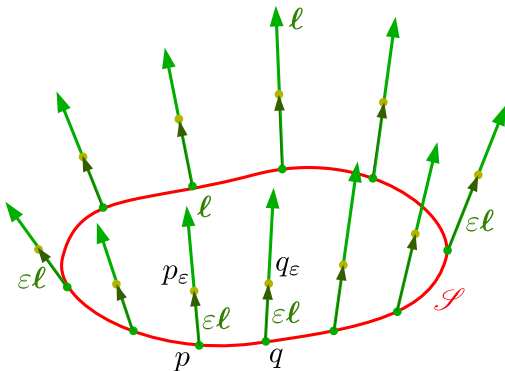
- 1 Consider a cross-section $\mathcal{S}$ and a null normal ℓ to $\mathcal{H}$

Expansion along a null normal



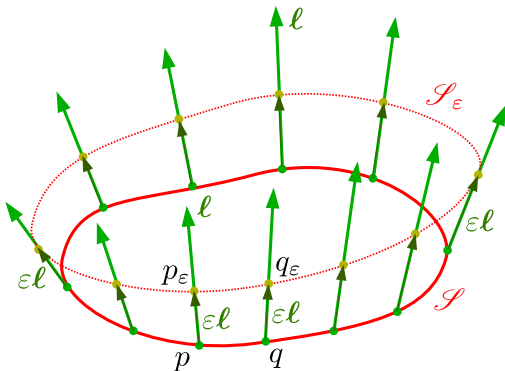
- 1 Consider a cross-section $\mathcal{S}$ and a null normal ℓ to $\mathcal{H}$
- 2 ε being a small parameter, displace the point p by the vector $\varepsilon \ell$ to the point p_ε

Expansion along a null normal



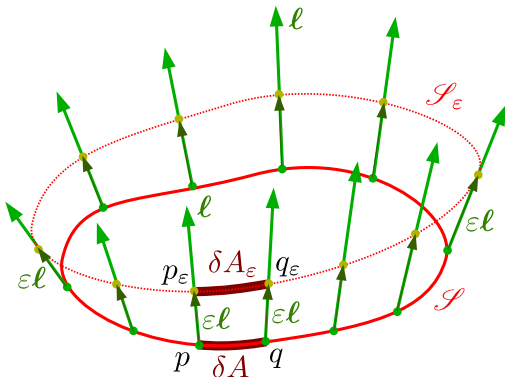
- 1 Consider a cross-section $\mathcal{S}$ and a null normal ℓ to $\mathcal{H}$
- 2 ϵ being a small parameter, displace the point p by the vector $\epsilon \ell$ to the point p_ϵ
- 3 Do the same for each point in $\mathcal{S}$, keeping the value of ϵ fixed

Expansion along a null normal



- 1 Consider a cross-section $\mathcal{S}$ and a null normal ℓ to $\mathcal{H}$
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- 3 Do the same for each point in $\mathcal{S}$, keeping the value of ε fixed
- 4 Since ℓ is tangent to $\mathcal{H}$, this defines a new cross-section $\mathcal{S}_\varepsilon$ of $\mathcal{H}$

Expansion along a null normal



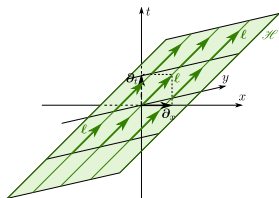
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The **expansion along ℓ** is defined from the relative change of the area δA (w.r.t. metric q) of a surface element δS of $\mathcal{S}$ around p :

$$\theta_{(\ell)} := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A} = \mathcal{L}_\ell \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu \ell_\nu$$

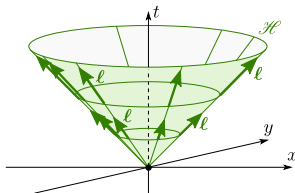
Examples of expansions

null hyperplane



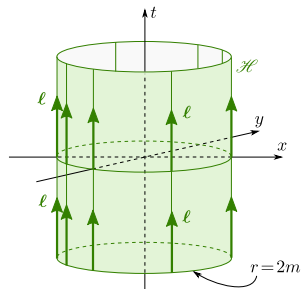
$$\theta_{(\ell)} = 0$$

future null cone



$$\theta_{(\ell)} = \frac{2}{r}$$

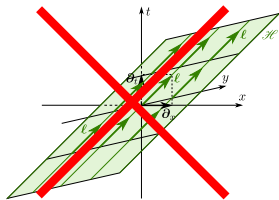
Schwarzschild horizon



$$\theta_{(\ell)} = 0$$

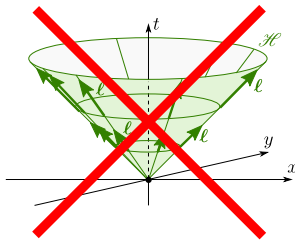
(Counter-)examples of non-expanding horizons

null hyperplane



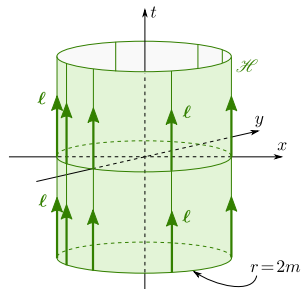
non-compact
cross-sections

future null cone



nonzero expansion

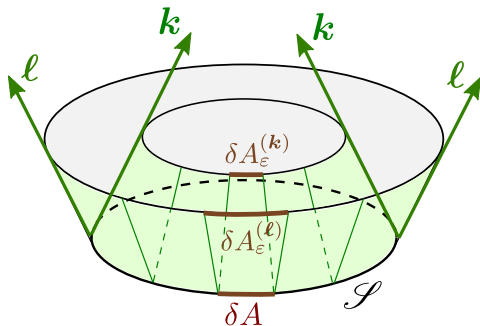
Schwarzschild horizon



OK

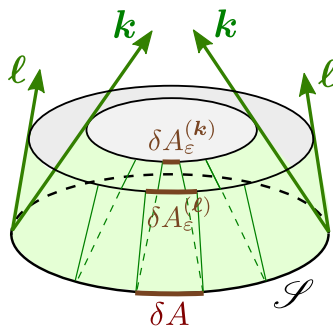
Trapped surfaces

untrapped surface



$$\theta_{(k)} < 0 \text{ and } \theta_{(\ell)} > 0$$

trapped surface



$$\theta_{(k)} < 0 \text{ and } \theta_{(\ell)} < 0$$

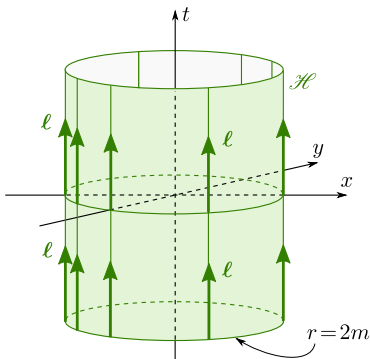
Example: area of the Schwarzschild horizon

Spacetime metric:

$$g = - \left(1 - \frac{2m}{r} \right) dt^2 + \frac{4m}{r} dt dr + \left(1 + \frac{2m}{r} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

$\mathcal{H}$: $r = 2m$; coord: (t, θ, φ)

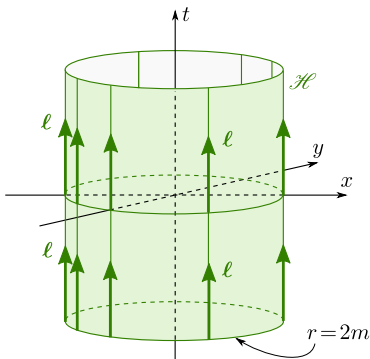
$\mathcal{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$



Example: area of the Schwarzschild horizon

Spacetime metric:

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$\mathcal{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathcal{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

$\Rightarrow$ induced metric on $\mathcal{S}$:

$$q_{ab} dy^a dy^b = (2m)^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

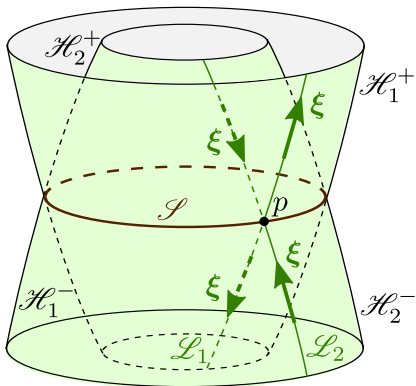
$$\Rightarrow q := \det(q_{ab}) = (2m)^4 \sin^2 \theta$$

$$\Rightarrow A = \int_{\mathcal{S}} (2m)^2 \sin \theta d\theta d\varphi$$

$$\Rightarrow \boxed{A = 16\pi m^2}$$

Bifurcate Killing horizons

$(\mathcal{M}, g) = n$ -dimensional spacetime endowed with a Killing vector field ξ



A **bifurcate Killing horizon** is the union

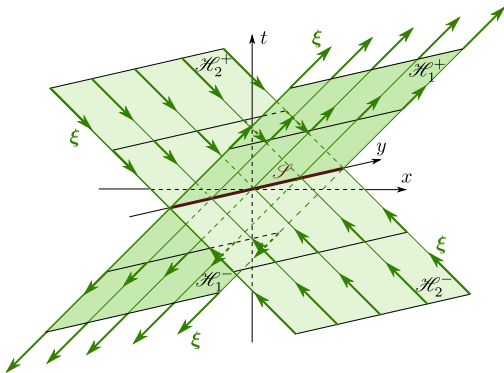
$$\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2,$$

where

- $\mathcal{H}_1$ and $\mathcal{H}_2$ are two null hypersurfaces;
- $\mathcal{I} := \mathcal{H}_1 \cap \mathcal{H}_2$ is a spacelike $(n - 2)$ -surface;
- each of the sets $\mathcal{H}_1 \setminus \mathcal{I}$ and $\mathcal{H}_2 \setminus \mathcal{I}$ has two connected components, which are Killing horizons w.r.t. ξ .

The $(n - 2)$ -dimensional submanifold $\mathcal{I}$ is called the **bifurcation surface** of $\mathcal{H}$.

Example 1: bifurcate Killing horizon w.r.t. a Lorentz boost generator



$(\mathcal{M}, g)$: Minkowski spacetime

$$g = -dt^2 + dx^2 + dy^2 + dz^2$$

Killing vector: $\xi = x\partial_t + t\partial_x$

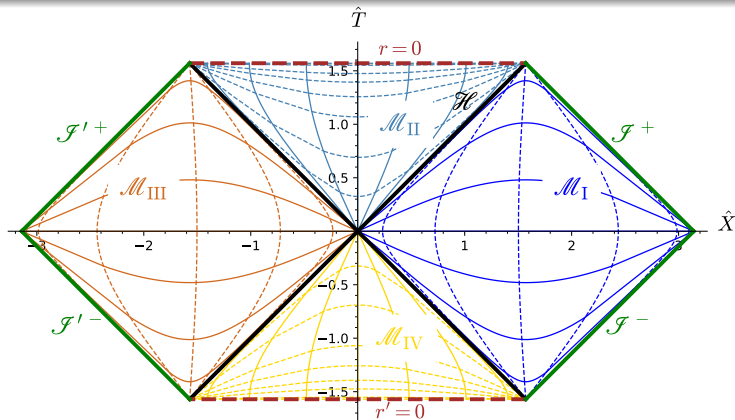
$\Rightarrow$ generates Lorentz boosts in the plane (t, x)

$$\mathcal{H}_1: t = x$$

$$\mathcal{H}_2: t = -x$$

$$\mathcal{S}: (t, x) = (0, 0)$$

Example 2: bifurcate Killing horizon in Schwarzschild spacetime

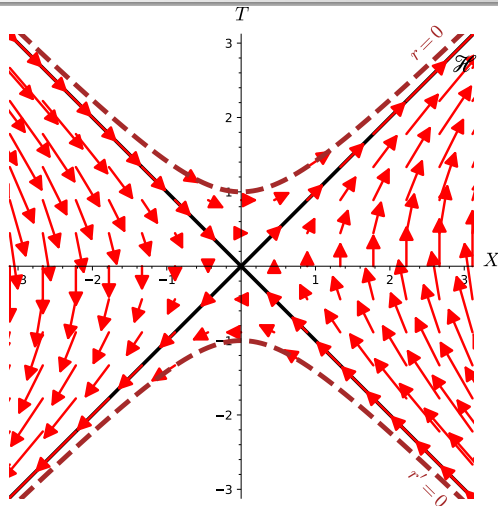


Dashed lines: field lines of the stationary Killing vector ξ

Thick black lines: bifurcate Killing horizon w.r.t. ξ

https://nbviewer.org/github/egourgoulhon/BHlectures/blob/master/sage/Schwarz_conformal_std.ipynb

Example 2: bifurcate Killing horizon in Schwarzschild spacetime



Stationary Killing vector ξ
in the maximal extension of
Schwarzschild spacetime
(Kruskal diagram)

https://nbviewer.org/github/egourgoulhon/BHlectures/blob/master/sage/Schwarz_

[Kruskal_Szekeres.ipynb](#)

Éric Gourgoulhon (LUX)

Introduction to black hole theory 2

ENS, Paris, 7 May 2025

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